

A Micro-Founded Mechanism For Prudential Decision Rule

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Abstract

In the aftermath of the Global Financial Crisis, both micro- and macro-prudential regulation are at work, but they are based on "ad-hoc" rules. This paper proposes an alternative micro-founded optimal prudential rule. Our unique prudential regulation progressively conducts all banks to an optimal asset level, thus avoiding the "too big to fail" paradigm. For a given critical asset level up to which the PA can successfully constrain banks, the optimal asset level is shown to be equal to half this critical value. Our optimal policy rule depends on banks' size. The PA should hence constrain large banks to progressively reduce their assets level and conversely encourage small banks to progressively increase it. A simultaneous regulation of banks on different loan markets, which accounts for correlation risk, accelerates the speed of convergence toward bank system equilibrium whatever the initial level of the asset. This micro-founded mechanism can be easily implemented in Dynamic Stochastic General Equilibrium (DSGE) models designed for prudential policy analysis.

Keywords: Prudential Regulation, Micro-Founded Policy Rule, Correlation Risk Effect, Priors

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1 Introduction

The Global Financial Crisis has emphasized the limits of the financial regulation to deal with financial instability. Governments needed to bail out several large banks in order to avoid systemic crisis in financial markets. This confirms the importance of the issue about the impunity degree of large banks. As explained by **Pontell *et al.*, (2014)**, investigations have shown that the financial crisis came from of their riskiness behaviour. Even if authorities have left Lehman Brother bank collapse, they helped other banks to survive through massive indebtedness of governments and unconventional central bank policies (**Mishkin, 2011**). These aids were a signal of the governmental dependence on banking system.

The banking regulation was clearly not the best to insure financial stability. For the *pre* 2008 period, it was especially based on microprudential regulations (i.e. individual risk of each bank) whereas macroprudential risk effect of large banks wasn't taken account (**Brunnermeier *et al.*, 2009**). This lack of efficiency led financial authorities to quickly design new capital requirements through Basel 2.5 and Basel 3. These new regulations tried to manage macroprudential risk without taking into account spillover effects between banking activity (**Kashyap *et al.*, 2011**). New systemic macroprudential ratios were added for large banks and tighter control was imposed for them, but their efficiency seems different across countries (**Cosimano and Hakura, 2011**). Furthermore, there is an arbitrage risk for banks since Basel 3 allows banks to weight their assets differently. Indeed, an interbank loan is lighter weighted by regulation than a corporate loan since risk profile is different. Thus, spillover effect of a prudential policy is difficult to manage and leads to imbalances in several segments of banking sector.

Prudential regulations try to take up the challenge with empirical exercises such as stress test for macroprudential requirements (**Acharya *et al.*, 2014**) or statistical assumptions for microprudential requirements (Risk Weighted Asset follows a Gaussian law). But, both microprudential and macroprudential regulations are currently based on "ad-hoc" rules, without any theoretical micro-foundation.

This led to wonder what would be an efficient prudential policy and how to built it from a more theoretical point of view. Is the distinction between micro- and macro-prudential regulation really unavoidable?

To our knowledge, our paper is the first to address these questions and it provide a micro-founded mechanism for prudential decision rules. It suggests an alternative optimal prudential rule to the micro/macro-prudential regulation. The prudential risk is analysed through bank asset size (*Asset approach*) to get a tractable way to examine PA. The aim of the prudential policy would be to progressively conduct each bank towards an optimal asset level, thus avoiding some banks to become too big to fail and to destabilize the credit market. In our model, banks are allowed to act simultaneously on two loan markets. The PA can either regulate each loan market separately or simultaneously, by taking into account the correlation risk effect between the different loans.

When defining a critical asset level up to which the PA can successfully impose constraints to banks, the optimal situation corresponds to a banks' asset level that represents half the critical value. The micro-founded PA policy rule that insures the convergence of banks towards this optimal situation must take into account the particular size of banks. Thus, the PA would constrain big banks to progressively reduce their assets until the optimal size condition is verified, but it would encourage small banks to progressively increase towards the optimal asset level. By simultaneously taking into account the activity of banks on both segments of the credit market when deciding the optimal regulation, the PA can accelerate the convergence towards the optimal solution whatever the initial value of the asset size. This is so because the PA can coordinate its actions on the two loan markets in order to stabilize more quickly one of them, when a shock arises. We finally suggest a way to integrate the mechanism in a Dynamic Stochastic General Equilibrium (DSGE) model designed for prudential policy or policy-mix analyses. The rest of the paper is structured as follows. Section 2 is dedicated to the determination of the micro-founded optimal level of assets for banks, see half the maximum critical value fixed by the PA. It also checks the robustness of theoretical result with an empirical exercise on several banking data samples. Section 3 explains how a real PA should set up its prudential policy in order to progressively conduct banks towards the optimal solution previously obtained. It also analyses the convergence speed of the policy with and without correlation risk effect when a shock occurs on bank asset size in a given loan market. Section 4 explains the higher convergence speed in the presence of correlation risk effect and thus show the utility of the coordinated regulation of different loan markets by the PA. Section 5 introduces a simple way to incorporate the mechanism in a DSGE model by using the one provided by **Gerali *et al.*, (2010)**. Section 6 concludes.

2 The long-run equilibrium for prudential regulation

2.1 Theoretical framework

We are in a closed economy where a bank "i" acts in two loan markets to manage its asset quantities ($A_1(i)$ and $A_2(i)$ respectively). There is a Prudential Authority (henceforth PA) which regulates both sectors by implementing policy to limit asset level of banks individually. In our model, asset size is the main channel to represent banks risk behaviour which also corresponds to one of the mains Basel rules criteria to monitor the financial institutions.

The PA and the bank know that in each market there is a critical size up to which regulator can manage its size (also its risk). This threshold is denoted by $A_1(max)$ and $A_2(max)$ for market 1 and 2 respectively. This means that the authority can face this problem in one market whereas it is able to manage the risk level of bank in the other

market. Therefore, the worst scenario for authority would be to get this dilemma in both loan sectors. But, the more a bank is approaching the critical size on one market, the more it becomes "too big to fail" and feels less concerned by the authority constraint. In other words, for all asset level lower than the critical size, the bank remains under the control of the PA, but the PA capacity to control the bank is reducing when this later is approaching the critical value of assets.

We introduce heterogeneity in markets size through a scale parameter θ applied on these critical sizes:

$$A_1(max) = \theta A_2(max)$$

where θ represents the importance of the first loan market relative to the second one. This value is known both by PA and banks.

The aim of bank "i" would be to reach threshold asset level in both markets whichever the risk that this action would imply. However, it knows that the PA would not allow it. As a result, it decides to integrate the anticipated regulator action in its investment strategy. To do so, it sets up an adjusted multiplier of asset level allowed by the PA for each market. It proceeds in several steps¹ :

- First, since bank "i" observes the critical asset size in market 1, it defines the standard multiplier which would like to get in this market, without taking into account the risk :

$$\text{Standard Multiplier} = \frac{A_1(max)}{A_1(i)}$$

However, this multiplier has to be adjusted because bank knows that the PA prevents it from reaching the threshold level. Indeed, when deciding its policy, PA adjusts downward the previous bank standard multiplier by taking into account a *risk effect* denoted X . X is a private information of the PA and it will not be provided to the bank. Moreover, the PA constraint for bank i in market 1 is contingent to the size of the bank on the market 1 relative to its global size. Indeed, the more the bank is approaching the critical size and becomes "too big to fail" on market 1, the lower is the ability of the PA to constrain it. The PA knows that and takes it into account by imposing a weaker PA regulation. The *PA Rule* would thus imply an adjusted multiplier in market 1 given by:

$$\text{Adjusted Multiplier} = \frac{A_1(max)}{A_1(i)} e^{-X \left[\frac{A_1(max) - A_1(i)}{A_1(i) + A_2(i)} \right]}$$

Where $A_1(i) + A_2(i)$ represents the global asset size of bank "i".

¹We only describe the process in one market because it is identical for the other one.

- Second, the bank would like to know the expected value of the adjusted multiplier imposed by the PA to get its optimal asset size in the market 1. Nevertheless, X is to be defined in the range $[0, +\infty[$ which complicates computation to get the mean value. To get round this issue, we suggest to take the sum of each possible value of X and rescale the domain through a temporal index :

Consider the sum as the integral of all possible value of X in the interval $[0, +\infty[$ via the function $f(X)$:

$$f(X) = \int_0^{+\infty} \frac{A_1(max)}{A_1(i)} e^{-X \left[\frac{A_1(max) - A_1(i)}{A_1(i) + A_2(i)} \right]} dX$$

Then we get :

$$F(X) = \frac{A_1(i) + A_2(i)}{A_1(i)} \frac{1}{1 - \frac{A_1(i)}{A_1(max)}}$$

We suppose that the threshold $A_1(max)$ is always higher than $A_1(i)$ which allows to express the previous equation as a geometric series with a common ratio of $\frac{A_1(i)}{A_1(max)}$:

$$F(X) = \frac{A_1(i) + A_2(i)}{A_1(i)} \sum_{t=0}^{\infty} \left(\frac{A_1(i)}{A_1(max)} \right)^t$$

We observe that the t index of the geometric series reflects the level of area under the function $f(X)$. Therefore, the index is a rescaling variable of the function and we assume that it corresponds to a time-horizon of prudential policy. This choice is motivated by the fact that time index acts as uncertainty parameter when it increases, i.e. when it considers a large horizon in the future. This is in line with the prudential policy because the higher is time-horizon, the lower is the additional regulation due to a higher uncertainty of bank asset level in the future.

Furthermore, the bank knows that only a part of this sum will be kept by the PA. To express this idea, we postulate that the geometric series is finite which leads to get a t index bounded at time T :

$$F^T(X) = \frac{A_1(i) + A_2(i)}{A_1(i)} \sum_{t=0}^T \left(\frac{A_1(i)}{A_1(max)} \right)^t$$

Equivalently, we have :

$$F^T(X) = \frac{A_1(i) + A_2(i)}{A_1(i)} \left[\frac{1 - \left(\frac{A_1(i)}{A_1(max)} \right)^{T+1}}{1 - \frac{A_1(i)}{A_1(max)}} \right]$$

- Third, bank "i" defines the optimal value of its assets in market 1. To do that, we compute the first order condition of $F^T(X)$ respect to $A_1(i)$.

$$\frac{\partial F^T(X)}{\partial A_1(i)} = 0$$

$$\frac{\left[1 - \left\{ \left(\frac{A_1(i)}{A_1(max)} \right)^{T+1} + \frac{(T+1)}{A_1(max)} (A_1(i) + A_2(i)) \left(\frac{A_1(i)}{A_1(max)} \right)^T \right\} \right] \left[A_1(i) - \frac{(A_1(i))^2}{A_1(max)} \right]}{\left[A_1(i) - \frac{(A_1(i))^2}{A_1(max)} \right]^2} - \frac{\left[1 - 2 \frac{A_1(i)}{A_1(max)} \right] \left[(A_1(i) + A_2(i)) \left(1 - \left(\frac{A_1(i)}{A_1(max)} \right)^{T+1} \right) \right]}{\left[A_1(i) - \frac{(A_1(i))^2}{A_1(max)} \right]^2} = 0$$

As we observe, when $A_1(i) = A_1(max)$, there is no defined solution but another one can be found if we consider at the same time that :

$$\left[1 - 2 \frac{A_1(i)}{A_1(max)} \right] = 0$$

$$A_1(i) = \frac{A_1(max)}{2} \tag{1}$$

And

$$\left[1 - \left\{ \left(\frac{A_1(i)}{A_1(max)} \right)^{T+1} + \frac{(T+1)}{A_1(max)} (A_1(i) + A_2(i)) \left(\frac{A_1(i)}{A_1(max)} \right)^T \right\} \right] = 0$$

$$(A_1(i) + A_2(i)) = \left[\frac{\left(\frac{A_1(max)}{A_1(i)} \right)^T - \frac{A_1(i)}{A_1(max)}}{(T + 1)} \right] A_1(max) \quad (2)$$

Regardless the time-horizon considered by the PA, equation (1) shows that optimal asset level for the bank in market 1 corresponds to the half of the critical size. This also reflects the best strategy of the bank when it has no information on the asset size expected by the authority.

Equation (2) describes the growth path of the bank global size (i.e., in both markets) with respect to time horizon. Since the first condition says that the optimal size of bank "i" in market 1 is independent of time horizon, equation (2) can be interpreted as the optimal growth path of bank "i" asset size in market 2.

If we inject solution (1) and (2) into $F^T(X)$, then we will obtain the optimal adjusted multiplier :

$$F^{opt-T}(X) = \frac{2^{T+2} + 2^{-T} - 4}{(T + 1)}$$

Where 2 stands for the optimal relative asset size $\frac{A_1(max)}{A_1(i)} = 2$ from equation (1). We see that without time-horizon (i.e. $T = 0$), the optimal multiplier becomes :

$$F^{opt-0}(X) = 1$$

As a result, if a PA implements a policy without time-horizon in its strategy and the bank is at the optimal asset size in market 1, then its optimal action will be to keep this size. For banks that have not the optimal size, in a convergence purpose towards the optimal size, the PA should opt to regulate differently large and small banks. Thus, large banks, whose assets on market 1 exceed the optimal size, should be constrained to come back to the optimum. Conversely, small banks should be encouraged to increase in order to gain the optimal size. More competition among banks of similar optimal size would simply be suitable for the financial stability.

2.2 Empirical evaluation of the T index value

Up to now, we have analysed bank behaviour without considering what is the scale of time index t for prudential policy horizon (day ?, week ?, month ?, quarter ?, etc.). To span this issue, we assume that a "t index" corresponds to a decade. This suggestion is based on time between changes in prudential regulation : 1987, Basel 1; 1999, Basel 2 even if it was implemented in 2004 and 2008 for Basel 3.²

²Bank for International Settlements, <https://www.bis.org/bcbs/history.htm>

In order to check the consistency of this suggestion, we propose to estimate an empirical relative asset size denoted k on several data samples. The aim is to see if results are close to the optimal value, i.e., when $k = 2 = k^{opt}$. To do so, we have to find the non-optimal form of adjusted multiplier which is possible if we assume that the condition of equation (1) is not satisfied. In this case, we can rewrite this equation by :

$$A_1(i) = \frac{A_1(max)}{k} \quad (3)$$

In economic terms, it means that bank "i" can't be to the optimal size immediately because of time to adjustment and other frictions (such as demand and supply ones). Moreover, the form of the equation (2) remain unchanged :

$$(A_1(i) + A_2(i)) = \left[\frac{\left(\frac{A_1(max)}{A_1(i)} \right)^T - \frac{A_1(i)}{A_1(max)}}{(T + 1)} \right] A_1(max)$$

Then we inject equation (2) and (3) into $F^T(X)$ to get the non-optimal form of the adjusted multiplier :

$$F^{T-No-opt}(X) = \frac{k^{T+2} + k^{-T} - 2k}{(k - 1)} \frac{1}{(T + 1)} \quad (4)$$

To estimate the empirical k , we have to proceed in several steps :

- a) We collect data samples from the Federal Reserve Bank of Saint Louis data base. We need long data to test for robustness since we look at decades. To do that, we take US banking series available from 1947 to 2017 : Loans and Lease, Commercial and Industrial Loans, Real Estate Loans and Consumer Loans.^{3 4}
- b) We suppose a representative bank whose allowed growth corresponds to the empirical multiplier. The later is computed as the average value of the upper / lower bound ratio of datas from a rolling window sample containing i decades observations. Since we have seven decades, we are able to calculate six empirical multipliers ⁵
- c) We estimate multipliers provided by our model through equation (4). The aim is to find the k which minimizes the gap between empirical and theoretical multipliers for each decade.

³European data only start from 1998 in ECB data base which is not convenient to test the robustness of the model (only two data retained).

⁴Data are seasonally adjusted.

⁵The seventh empirical multiplier is not computed because we can't use the rolling window process.

Table 1: Estimated k from first to sixth decades

Period / Data	Loans and Lease	Commercial and Industrial Loans	Real Estate Loans	Consumer Loans
10 years	2.15	2.11	2.24	2.13
20 years	2.22	2.13	2.44	2.15
30 years	2.37	2.25	2.68	2.28
40 years	2.44	2.29	2.8	2.33
50 years	2.46	2.28	2.8	2.33
60 years	2.47	2.26	2.77	2.35

Table 1 shows estimated k of the four types of loan at each decade. Results indicates that estimated k are not too far from the optimal value ($k = 2$) since the higher one is equal to 2.8 and corresponds to real estate loans. This market provides higher results because of the boom mortgage credit before subprime crises.

We can resume our main result in this section as follows. Assuming that banks have incomplete information on the market and integrate in their decision rule their perception on the prudential policy, we were able to show that the optimal value provided the PA corresponds to the half of a threshold asset size $A^1(max)$ which is known by all agents in the market. To get this result, we have assumed the existence of temporal index which permits to get round the mean Riemann integral issue. The robustness of this idea was checked through an empirical exercise to examine if the optimal multiplier provided by the mechanism depart from the empirical multiplier of market loans.

3 The prudential policy mechanism

We have seen in the previous section that the optimal asset size of bank is reached when it corresponds to the half of the threshold level. The definition of an optimal asset level makes a converging prudential policy possible for the PA. This later may want to stabilize the credit sectors separately or simultaneously, by insuring a progressive convergence towards the optimal size. It decides to consider two scenarios in its prudential rule: with and without correlation between credit sectors. In absence of this correlation effect, the PA separately conducts its policy on each market. It only cares about the evolution of bank asset in each sector. However, when the correlation is incorporated in the PA decision rule, this one is simultaneously conduct policies on both markets. It is forced to consider the stability of all credit sectors rather than to take into account only

one. As we will see, this additional issue incites the PA to implement a faster convergent policy.

In this part, we come back to the PA policy rule and are going to see how the authority fixes the risk level X and which is the explicit form of the PA policy rule.

The PA has to respect two criteria in the conduct of its action : progressive and converging policy. The first feature characterizes the PA willingness to smooth its impact on the economy by avoiding sudden adjustment of banks. This is also in line with the *phase in process* of Basel rules to implement a policy⁶. The second objective describes the wish of authority to get homogeneity of bank size in a market in order to encourage the competition among them. In our model, the converging policy implies for the PA to define a benchmark asset level. Since it knows the optimal asset size self-defined by the bank in the previous part, it decides to consider $\frac{A_1(max)}{2}$ as the benchmark.

3.1 A prudential policy without correlation risk effect.

We keep the same economic environment as in the previous section : we are in a closed economy with two loans market where a PA regulates each bank "i" in their respective market. For simplicity, we will consider in this section that the two loan markets have similar size, which implies θ equal to 1. Such as before, we are going to study the construction of adjusted multiplier under prudential policy for one market with the assumption of symmetrical features for the set up of the multiplier in the other market.

We suppose that the PA implements the following prudential policy to build its adjusted multiplier :

$$\text{Adjusted multiplier} = \frac{A_1(max)}{A_1(i)} e^{-X \left[\frac{A_1(max) - A_1(i)}{A_1(i) + A_2(i)} \right]} \quad (5)$$

Contrary to the bank, PA decides the level of X which drives its prudential policy. It considers this variable as the sum of standard multipliers time-variance of both markets. However, it doesn't take into account the potential correlation between these two markets. Therefore, the adjusted multiplier defined by the PA can be rewritten as :

$$\text{Adjusted multiplier} = \frac{A_1(max)}{A_1(i)} e^{-[Var(k_1) + Var(k_2)] \left[\frac{A_1(max) - A_1(i)}{A_1(i) + A_2(i)} \right]} \quad (6)$$

⁶See for example the *phase in process* of the leverage ratio in Basel III which was fully implemented in 2018.

Where $Var(k_m) = Var\left(\frac{A_m(max)}{A_1(m)}\right)$ (for $m = 1, 2$) stands for the time-variance of standard multiplier in market m .

To define the consistent risk measure $Var(k_1) + Var(k_2)$, we assume that the PA has two priorities in the setting up of its prudential policy. In one hand, it wants to bound its regulation on a bank "i" in market 1 which has not reached the critical level since it has no control on its risk in that case. In the other hand, the PA supposes that there is no correlation between markets so it has to define the level of global time-variance which incites to only focus on evolution of assets in market 1 when the regulation is implemented in this market.

To answer these two issues, we suggest to proceed as follows :

1) Linearise with natural logarithm the adjusted multiplier denoted by $M(A_1(i))$:

$$\ln(M(A_1(i))) = -[Var(k_1) + Var(k_2)] \left[\frac{A_1(max) - A_1(i)}{A_1(i) + A_2(i)} \right] + \ln(A_1(max)) - \ln(A_1(i))$$

2) Express the first order condition of the previous equation w.r.t $A_1(i)$:

$$\begin{aligned} \frac{\partial \ln(M(A_1(i)))}{\partial A_1(i)} &= 0 \\ -[Var(k_1) + Var(k_2)] \left[\frac{-(A_1(i) + A_2(i)) - (A_1(max) - A_1(i))}{(A_1(i) + A_2(i))^2} \right] - \frac{1}{A_1(i)} &= 0 \end{aligned}$$

3) Consider $A_1(i) = A_1(max)$ and deduce that :

$$Var(k_1) + Var(k_2) = \frac{A_1(i) + A_2(i)}{A_1(max)} \quad (7)$$

However we look at a functional form to get global time-variance for each value of $A_1(i)$ and not only when $A_1(i) = A_1(max)$. Since we assume that there is no correlation between two markets, we suggest to keep the form $Var(k_1) + Var(k_2) = \frac{\sum_{m=1}^2 A_m(i)}{A_{1;2}(max)}$ which cancels out the effect of the market 2 assets evolution in the prudential regulation of assets in market 1.

Therefore, for any value of $A_1(i)$, global time-variance can be expressed as :

$$\frac{A_1(i) + A_2(i)}{A_1(max)} = \left(\frac{A_1(i)}{A_1(max)} + \frac{A_2(i)}{A_1(max)} \right)$$

Through the market size identity we get⁷ :

$$\frac{A_1(i) + A_2(i)}{A_1(max)} = \left(\frac{A_1(i)}{A_1(max)} + \frac{\theta A_2(i)}{A_2(max)} \right)$$

Consequently, in the absence of correlation and by injecting equation (7) into (6), the adjusted multiplier defined by the PA is :

$$\text{Adjusted multiplier PA} = \frac{A_1(max)}{A_1(i)} e^{-\left(1 - \frac{A_1(i)}{A_1(max)}\right)}$$

Moreover, the PA wants a converging prudential policy toward the benchmark asset size $\frac{A_1(max)}{2}$. To do that, it integrates a converging coefficient γ in the previous formula⁸ :

$$\text{Adjusted multiplier PA} = \frac{1}{\gamma} \frac{A_1(max)}{A_1(i)} e^{-\gamma \left(1 - \frac{A_1(i)}{A_1(max)}\right)}$$

Equivalently :

$$\text{Adjusted multiplier PA} = \frac{1}{\gamma} k_1 e^{-\gamma \left(1 - \frac{1}{k_1}\right)}$$

Where $k_1 = \frac{A_1(max)}{A_1(i)}$.

Once the adjusted multiplier has been defined by the PA, one can compute the maximum relative asset size allowed by the PA. It can be considered as a *ceiling value* that bank "i" is not allowed to exceed and it is denoted by $k_1^{PA-ceiling}$.

$$k_1^{PA-ceiling} = \frac{A_1(max)}{A_1(ceiling)} = \frac{A_1(max)}{\text{Adjusted multiplier PA} \times A_1(i)} = \frac{\gamma A_1(max)}{k_1 e^{-\gamma \left(1 - \frac{1}{k_1}\right)} \times A_1(i)}$$

We can simplify by :

$$k_1^{PA-ceiling} = \gamma e^{\gamma \left(1 - \frac{1}{k_1}\right)}$$

Since there is no frictions in our mechanism, the value of k_1 is automatically adjusted to the $k_1^{PA-ceiling}$ required by the PA⁹. As it is shown in the following graph, the convergence to $k_1 = 2$ is guaranteed whatever its initial value of k_1 . Figure 1 simulates the

⁷For simplicity, we assume that both loans market have the same size, i.e., $\theta = 1$.

⁸Due to the non-linearity, the value of γ is found by a solver which satisfies the following condition $\gamma e^{0.5\gamma} = 2$. Solution gives $\gamma \approx 1.1343$.

⁹In our simulation, this assumption is equivalent to say that $k_m^{PA-ceiling}$ corresponds to the k_1 at the next period.

dynamic path of $k_1^{PA-ceiling}$ with an initial value of $k_1 = 5$ and we observe that it needs almost ten periods to converge toward the optimal value properly.

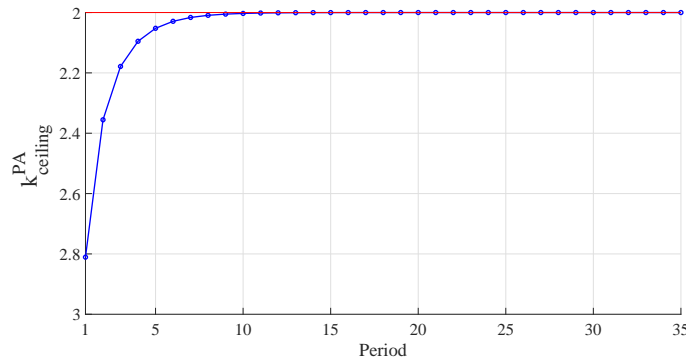


Figure 1: Dynamic path of $k_{ceiling}^{PA}$ (initial value $k_1 = 5$).

This type of convergence speed can be appreciated in DSGE models since these ones look at short term modifications of the economy after a shock. Indeed, if we start our dynamic model with $k_1 = 2$ and put a negative deterministic shock to reduce bank asset size of 10% ($k_1 = 2.22$) at the fifteenth period, then the convergence time will be relatively close to ten periods again (see figure 2). This type of dynamic can be useful to mimic gradual and non-linear changes of PA and bank behaviour close to the optimal asset size.

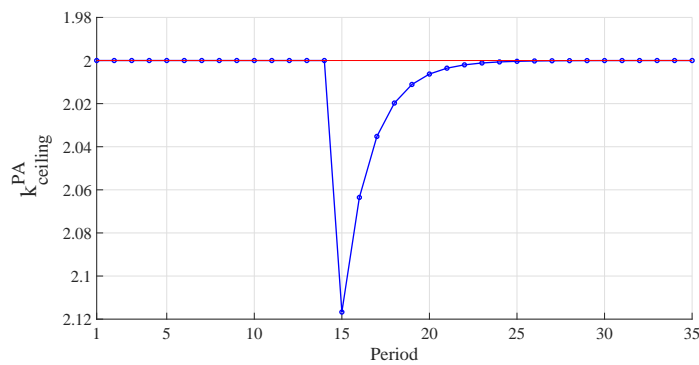


Figure 2: Impact of asset size negative shock on $k_{ceiling}^{PA}$ dynamic path.

Furthermore, the convergence speed seems similar for a positive or negative shock without other frictions (such as demand or supply constraints). We demonstrate that by simulating the same negative shock plus a positive one to increase bank asset size of 10% ($k_1 = 1.82$) at the thirty-fifth period. Figure 3 draws the dynamic path and we notice

that positive shock disappears after ten periods such as negative one. We will see in next parts that the integration of the correlation risk effect influences the convergence speed.

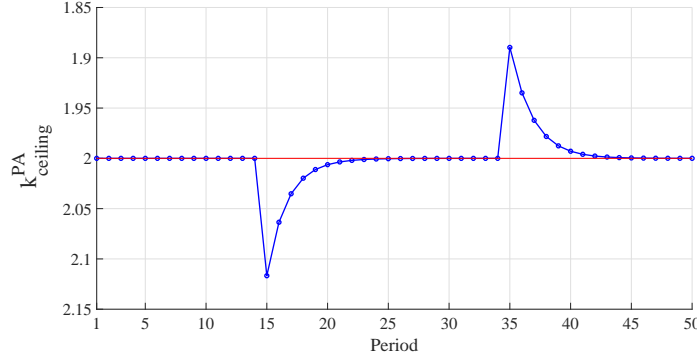


Figure 3: Impact of asset size negative and positive shock on $k_{ceiling}^{PA}$ dynamic path.

3.2 A prudential policy with correlation risk effect

In this part, we extend the prudential regulation of the PA by integrating the correlation in global time-variance. This seems a realistic feature since the authority guesses that its regulation on one loan market affect the prudential management in the other market. The underlying consequences comes from the reaction of bank to smooth the prudential regulation cost by adjusting the level of its two asset types.

As a result, authority is not able to define the level of correlation which depends of bank strategies. These strategies is not communicated by the bank because it will become public informations and provide market power for bank competitors. An other reason is the lack of skill for PA to gauge the correlation properly. Empirical evidences show that prudential regulators suggest banks to assess themselves the riskiness degree of their asset (*IRB and IRB advanced model*) even if authorities provide an external tool of risk assessment (*Standard approach*). The aim for regulators is to better understand specificities of some banking sectors (e.g, correlation targeting for portfolio strategies, clients management,...) to improve their own risk evaluation model.

To reflect this idea, we suggest to consider a PA which not evaluates properly the global time-variance because of the correlation effect. The level of this effect is an unknown variable Y for authorities which leads to define its adjusted multiplier as :

$$\text{Adjusted multiplier PA} = \frac{A_1(max)}{A_1(i)} e^{-(Var(k_1)+Var(k_2)+2\sigma_{k_1}\sigma_{k_2}Y) \left[\frac{A_1(max)-A_1(i)}{A_1(i)+A_2(i)} \right]}$$

Where σ_{k_1} and σ_{k_2} stands for the standard deviation of standard multiplier in market 1 and 2 respectively.

Since it is an unknown, the PA takes the expected value of its adjusted multiplier denoted by $f_1(Y)$:

$$f_1(Y) = \frac{1}{2} \int_{-1}^1 \frac{A_1(max)}{A_1(i)} e^{-(Var(k_1)+Var(k_2)+2\sigma_{k_1}\sigma_{k_2}Y) \left[\frac{A_1(max)-A_1(i)}{A_1(i)+A_2(i)} \right]} dY$$

After some algebra computations, we can express the expected adjusted multiplier of PA as :

$$f_1(Y) = \frac{A_1(max)}{A_1(i)} e^{-(Var(k_1) + Var(k_2))g(A_m(i))} \prod_{j=0}^{\infty} \cosh \left(\left(\frac{\sigma_{k_1}\sigma_{k_2}}{2^j} \right) g(A_1(i)) \right) \quad (8)$$

Where $g(A_1(i)) = \left[\frac{A_1(max) - A_1(i)}{A_1(i) + A_2(i)} \right]$ and \cosh corresponds to the hyperbolic cosine.

Notice that the correlation term is caught by the Cartesian product of the hyperbolic cosine¹⁰. The more multiplicative terms j is added, the wider the range of correlation is considered by the PA. If we take the correlation effect with only $j = 0$, then it will be equivalent to say that the PA scans the expected value for the case of either $correlation = 0.5$ or $correlation = -0.5$ (see figure 4). If we take a larger correlation risk effect range, for instance, until $j = 2$, then it will take into account an additional "correlation area" respect to the initial case $j = 0$. However, the increase of this area tends to reduce when j goes to $+\infty$ which means that the PA is less concerned by the contribution of an additional increase of the correlation range. Figure 5 shows that contribution is especially based on the four first j terms of the Cartesian product which indicates a strong decrease of the marginal interest for an additional correlation area.¹¹ To define the consistent risk measure $Var(k_1) + Var(k_2)$, we assume that the PA has two priorities. In one hand, it wants to bound its regulation on a bank "i" in market 1 which has not reached the critical size since it can't manage its risk in that case. In the other hand, we suppose that the correlation has only a multiplicative effect on the adjusted multiplier without correlation defined previously.

To answer these two issues, we suggest to proceed as follows :

¹⁰This hyperbolic property permits to get round the undefined solution problem in the definition of global time-variance in next paragraphs.

¹¹ σ_{k_1} and σ_{k_2} are defined by the equation (8) and (9) of the next part while γ^{correl} is obtained via a solver (see p.21).

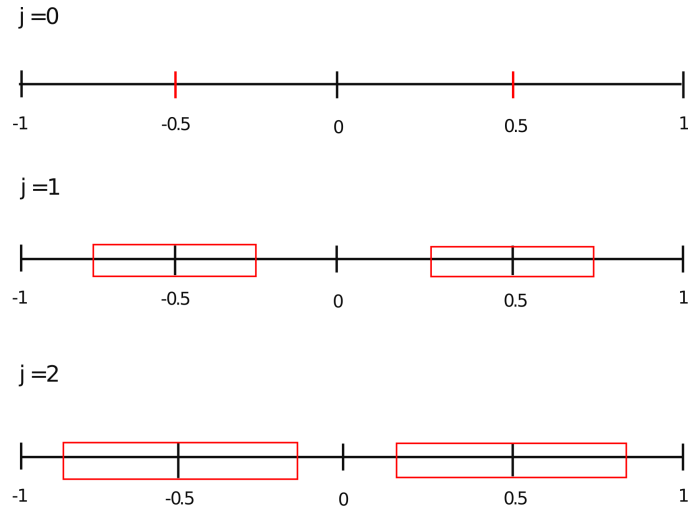


Figure 4: Range of correlation scanned by the PA.

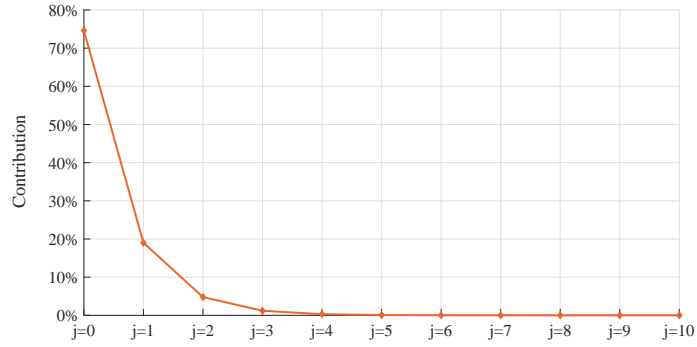


Figure 5: Marginal log-contribution of a range correlation increase.

1) Linearise with natural logarithm the expected adjusted multiplier :

$$\ln(f_1(Y)) = -[Var(k_1) + Var(k_2)] \left[\frac{A_1(max) - A_1(i)}{A_1(i) + A_2(i)} \right] + \ln(A_1(max)) - \ln(A_1(i))$$

$$+ \sum_{j=0}^{\infty} \ln \left(\cosh \left(\left(\frac{\sigma_{k_1} \sigma_{k_2}}{2^j} \right) g(A_1(i)) \right) \right)$$

2) Express the first order condition of the previous equation w.r.t $A_1(i)$:

$$\frac{\partial \ln(f_1(Y))}{\partial A_1(i)} = 0$$

$$\begin{aligned}
& -[Var(k_1) + Var(k_2)] \left[\frac{-(A_1(i) + A_2(i)) - (A_1(max) - A_m(i))}{(A_1(i) + A_2(i))^2} \right] - \frac{1}{A_m(i)} \\
& + \sigma_{k_1} \sigma_{k_2} \left[\frac{(A_1(i) + A_2(i)) - (A_1(max) - A_1(i))}{(A_1(i) + A_2(i))^2} \right] \sum_{j=0}^{\infty} \frac{1}{2^j} \tanh \left(\left(\frac{\sigma_{k_1} \sigma_{k_2}}{2^j} \right) g(A_1(i)) \right) = 0
\end{aligned}$$

Where $\tanh()$ stands for the hyperbolic tangent.

3) Consider $A_1(i) = A_1(max)$ and deduce that :

$$Var(k_1) + Var(k_2) = \frac{(A_1(i) + A_2(i))}{A_{1;2}(max)} \quad (9)$$

To get a tractable function, we make the hypothesis that the correlation risk effect has only a multiplicative effect on PA decision rule. This assumption leads to get the functional form of $Var(k_1) + Var(k_2)$ for any $A_1(i)$:

$$Var(k_1) + Var(k_2) = \frac{(A_1(i) + A_2(i))}{A_1(max)} \quad \forall A_1(i)$$

Furthermore, the PA seeks to homogenize banks size in market 1 by converging them toward the benchmark level $\frac{A_1(max)}{2}$. To do so, authority integrates in the same way as the previous part a converging coefficient γ^{correl} in its expected adjusted multiplier :

$$\text{Adjusted multiplier PA} = \frac{1}{\gamma^{correl}} k_1 e^{-\gamma^{correl}} \left(1 - \frac{1}{k_1}\right) \prod_{j=0}^{\infty} \cosh \left(\left(\frac{\sigma_{k_1} \sigma_{k_2}}{2^j} \right) g(A_1(i)) \right)$$

In the previous formula, we have defined all parameters except standard deviation for both markets. Hence, we start from the assumption made on the global time-variance :

$$Var(k_1) + Var(k_2) = \left(\frac{A_1(i) + A_2(i)}{A_1(max)} \right)$$

To get standard deviations, we have to define the value of each variances. The problem is we only have one equation for two unknown. We assume that variance in market 1 is linked to the other one in market 2 such as it corresponds to a linear combination of $\frac{1}{k_1}$ and $\frac{1}{k_2}$. Furthermore, we add two propositions to define each of them.

Proposition 1. *Proportional distribution of contributions* : Since the sum of the two

variances have to satisfy the equation above, the proportional distribution of contributions for k_1 and k_2 can be defined by¹²:

$$\begin{aligned} Var(k_1) &= \left(\alpha_1 \frac{1}{k_1} + (1 - \alpha_2) \frac{1}{\theta k_2} \right) \\ Var(k_2) &= \left((1 - \alpha_1) \frac{1}{k_1} + \alpha_2 \frac{1}{\theta k_2} \right) \end{aligned}$$

Proposition 2. *Proportional contribution in variance* : The PA reasons with proportional contributions which means that it defines the contribution of k_1 and k_2 for one variance. Due to the proportionality, the sum of two contributions has to be equal to one. Moreover, the proposition 1 leads the PA to provides a symmetrical set of contributions for the other variance. As a result, $\alpha_1 = \alpha_2 = \alpha$ and both variances defined by the PA can be rewritten as follows¹³ :

$$\begin{aligned} Var(k_1) &= \left(\alpha \frac{1}{k_1} + (1 - \alpha) \frac{1}{\theta k_2} \right) \\ Var(k_2) &= \left((1 - \alpha) \frac{1}{k_1} + \alpha \frac{1}{\theta k_2} \right) \end{aligned}$$

Notice that if PA considers $\alpha \neq 1$, then it will admit the existence of correlation between two markets. However this level of correlation corresponds to the PA personal opinion since this later wants to evaluate the correlation between two types of asset properly and only bank "i" can do that.

We are now able to express standard deviations :

$$\sigma_{k_1} = \sqrt{\left(\alpha \frac{1}{k_1} + (1 - \alpha) \frac{1}{\theta k_2} \right)} \quad (10)$$

$$\sigma_{k_2} = \sqrt{\left((1 - \alpha) \frac{1}{k_1} + \alpha \frac{1}{\theta k_2} \right)} \quad (11)$$

¹²Notice that relative market size impact θ is analysed from the market 1 prudential regulation perspective. From market 2 side, we will get $Var(k_1) = \left(\alpha_1 \frac{\theta}{k_1} + (1 - \alpha_2) \frac{1}{k_2} \right)$ and $Var(k_2) = \left(\alpha_1 \frac{\theta}{k_1} + (1 - \alpha_2) \frac{1}{k_2} \right)$. Since in our simulation $\theta = 1$, this does not affect results.

¹³If we neglect proposition 2, then the PA will not reason in proportional contribution and other non-linear effects will occur. For simplicity, these features will be studied in a future framework

Consequently, the expected adjusted multiplier corresponds to :

$$f_1(Y) = \frac{\frac{k_1}{\gamma^{correl}} e^{-\gamma^{correl} \left(1 - \frac{1}{k_1}\right)}}{\left[\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha \frac{1}{k_1} + (1-\alpha) \frac{1}{\theta k_2}} \sqrt{(1-\alpha) \frac{1}{k_1} + \alpha \frac{1}{\theta k_2}} \right) g(k_1) \right) \right]^{-1}}$$

With :

$$g(A_1(i)) = \left[\frac{A_1(max) - A_1(i)}{A_1(i) + A_2(i)} \right]$$

$$g(A_1(i)) = \left[\frac{1 - \frac{1}{k_1}}{\frac{1}{k_1} + \frac{1}{k_2}} \right] = g(k_1)$$

The parameter γ^{correl} corresponds to the converging coefficient applied by the authority¹⁴.

Once all parameters defined, we can compute the maximum relative asset size allowed by the PA, i.e. the ceiling value $k_1^{PA-ceiling}$.

$$k_1^{PA-ceiling} = \frac{\gamma^{correl} e^{\gamma^{correl} \left(1 - \frac{1}{k_1}\right)}}{\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha \frac{1}{k_1} + (1-\alpha) \frac{1}{\theta k_2}} \sqrt{(1-\alpha) \frac{1}{k_1} + \alpha \frac{1}{\theta k_2}} \right) g(k_1) \right)}$$

Figure 6 draws the convergent dynamic path of $k_1^{PA-ceiling}$ with correlation effect (dashed line) and without it (solid-dotted line) when initial value of relative asset size is equal to $k_1 = 5$ and $k_2 = 2$ in market 1 and 2 respectively¹⁵.

We observe that the correlation risk effect accelerates the convergence toward the optimal asset level compared to standard case. Since the PA knows that its regulation in market 1 affects its prudential management in the other market, it decides to limit this impact by accelerating the convergence toward the optimal asset level in market 1.

3.3 Negative shock on $A_{1t}(i)$

Figure 7 shows the effect of a negative shock which reduces of 10% bank size in market 1 on its $k_1^{PA-ceiling}$ level. We observe that the shock leads to reduce the ceiling rule of

¹⁴The value of γ^{correl} has to satisfy the convergence condition : $\gamma^{correl} e^{\gamma^{correl} \left(1 - \frac{1}{2}\right)} \left[\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha \frac{1}{2} + (1-\alpha_2) \frac{1}{2}} \sqrt{(1-\alpha_1) \frac{1}{2} + \alpha_2 \frac{1}{2}} \right) g(2) \right) \right]^{-1} = 2$

¹⁵For simplicity, we don't consider dynamic paths of the rule where $\alpha \neq 1$ since they provide same shapes except for $\alpha = 0.5$ but this particular case will be analysed in a next part.

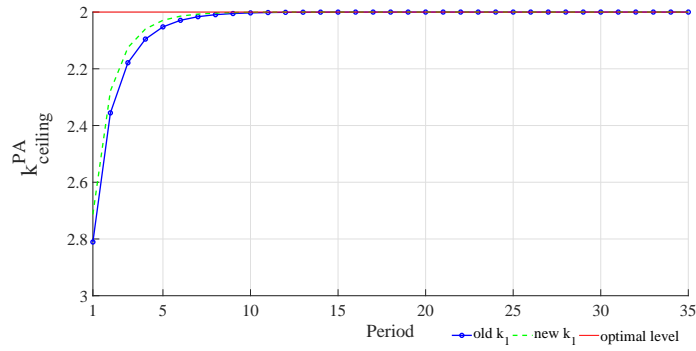


Figure 6: Dynamic path of $k_1^{PA-ceiling}$ with correlation risk effect (initial value $k_1 = 5$, $k_2 = 2$, $\alpha = 1$).

$k_1^{PA-ceiling}$ at almost 2.12. This decreasing is less pronounced than the situation without correlation risk effect (see solid-dotted line). The reason has been mentioned before : the PA wants to limit the impact of its market 1 prudential policy on the market 2.

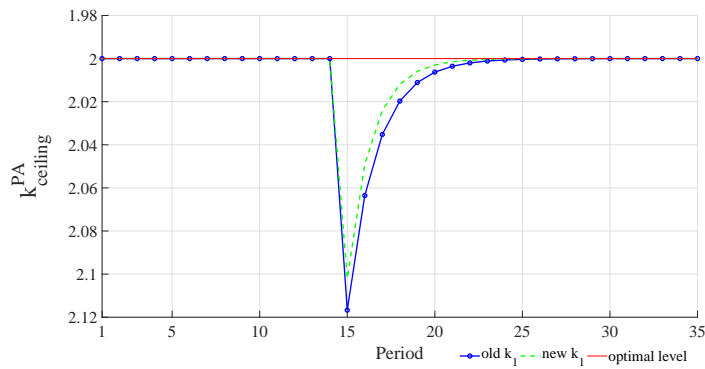


Figure 7: Impact of a negative asset 1 size shock with $\alpha = 1$

3.4 Positive shock on $A_{1t}(i)$

Figure 8 depicts the effect of a positive shock which increases of 10% bank size in market 1 on its $k_1^{PA-ceiling}$ level. Results confirm the idea mentioned above since the PA accelerates the convergence of its market 1 prudential regulation to limit its impact on market 2.

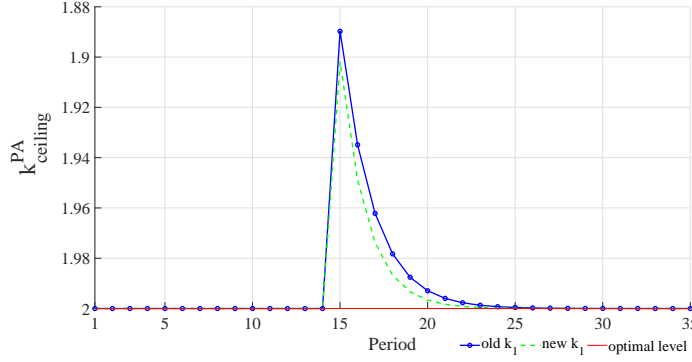


Figure 8: Impact of a positive asset 1 size shock with $\alpha = 1$

4 Analysis of the dynamic PA behaviour

4.1 Elasticity analysis of bank income

This part studies the impact of a bank asset level change on the prudential regulation. To do so, we consider the classical banking income function $\rho(i)$:

$$\rho(i) = R_t^{A_1} A_{1t}(i) + R_t^{A_2} A_{2t}(i)$$

Where $R_t^{A_1}$ and $R_t^{A_2}$ stand for the interest rates applied in loan market 1 and 2 respectively. We assume that loan supply satisfies the demand and if bank decides to increase a marginal amount of credit, it will match with a borrower. Prudential regulation is applied as a regulation of credit supply which means that the new banking income function $\rho(i)$ integrates the asset level allowed by the PA in each market $A_{1t}^{ceiling}(i)$ and $A_{2t}^{ceiling}(i)$.

Hence :

$$\rho^{prudential}(i) = R_t^{A_1} A_{1t}^{ceiling}(i) + R_t^{A_2} A_{2t}^{ceiling}(i)$$

4.1.1 Without correlation risk effect

We consider the prudential rule without correlation risk effect to analyse the elasticity between bank "i" and PA behaviour. This elasticity in market "m" can be expressed as :

$$e_m(i) = \frac{\partial A_{mt}^{ceiling}(i)}{\partial A_{mt}(i)} \times \frac{\bar{A}_{mt}(i)}{\bar{A}_{mt}^{ceiling}(i)}$$

Where $\bar{A}_{mt}(i)$ and $\bar{A}_{mt}^{ceiling}(i)$ are the steady-state value of bank "i" and PA-ceiling asset level respectively. Since we have supposed that $A_{mt}(i)$ corresponds to $A_{mt}^{ceiling}(i)$ with a one-period-lag in simulations, we deduce that $\bar{A}_{mt}(i) = \bar{A}_{mt}^{ceiling}(i)$. Therefore, the

elasticity corresponds to the marginal effect of the ceiling rule in market "m" when bank "i" increases its asset :

$$e_m(i) = \frac{\partial A_{mt}^{ceiling}(i)}{\partial A_{mt}(i)}$$

To get the value of $A_{mt}^{ceiling}(i)$, we consider the ceiling rule provided by the PA for market 1 and 2 :

For market 1 :

$$\begin{aligned} \frac{A_1(Max)}{A_{1t}^{ceiling}(i)} &= \gamma e^{\gamma \left(1 - \frac{A_{1t}(i)}{A_1(Max)}\right)} \\ A_{1t}^{ceiling}(i) &= \frac{1}{\gamma} e^{-\gamma \left(1 - \frac{A_{1t}(i)}{A_1(Max)}\right)} A_1(Max) \end{aligned}$$

For market 2 :

$$\begin{aligned} \frac{A_2(Max)}{A_{2t}^{ceiling}(i)} &= \gamma e^{\gamma \left(1 - \frac{A_{2t}(i)}{A_2(Max)}\right)} \\ A_{2t}^{ceiling}(i) &= \frac{1}{\gamma} e^{-\gamma \left(1 - \frac{A_{2t}(i)}{A_2(Max)}\right)} A_2(Max) \end{aligned}$$

Hence, the elasticity for each market can be expressed as :

For market 1 :

$$e_1(i) = \gamma e^{-\gamma \left(1 - \frac{A_{1t}(i)}{A_1(Max)}\right)}$$

For market 2 :

$$e_2(i) = \frac{1}{\gamma} e^{-\gamma \left(1 - \frac{A_{2t}(i)}{A_2(Max)}\right)}$$

Figure 9 (figure 10) draws the dynamic path of the elasticity when a positive (negative) shock increases (decreases) of 10% bank asset size at the fifteenth period in market "m". Notice that before the shock, the PA provides a marginal asset level lower than the one expected by the bank "i" (which is equal to 1 by definition). This means that the PA implements a policy to not incite bank "i" to reach the critical size. When the shock occurs, the elasticity increases (decreases) because the PA sets up a progressive and converging prudential regulation to avoid sudden adjustment of bank which would lead to a high volatility in the market. Nevertheless, the prudential regulation still constrains

bank "i" since the elasticity remains below the unity after the shock. Consequently, in a situation where the interest is fixed, the marginal income obtained by the bank "i" under prudential regulation is lower than the one it should get without policy :

$$\frac{\partial \rho^{new}(i)}{\partial A_{mt}(i)} < \frac{\partial \rho(i)}{\partial A_{mt}(i)}$$

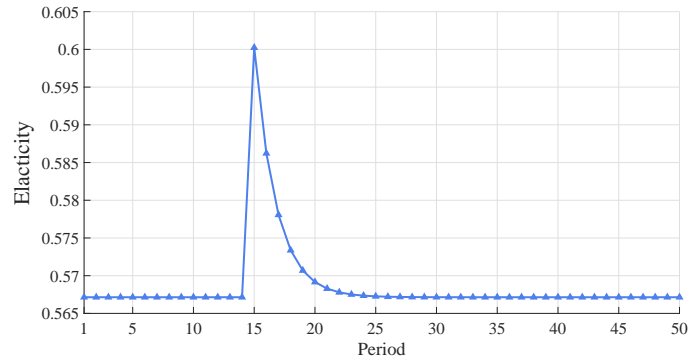


Figure 9: Impact of a 10% asset size increase on the elasticity.

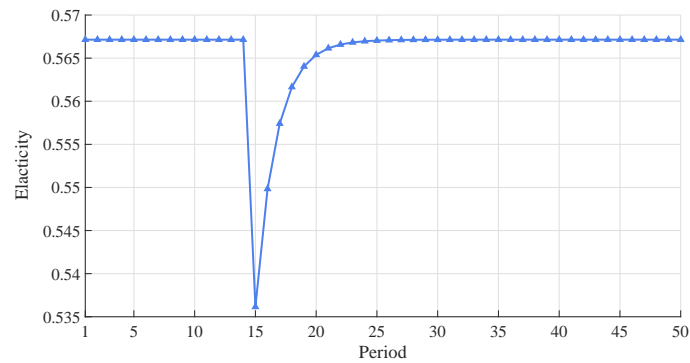


Figure 10: Impact of a 10% asset size decrease shock on the elasticity.

4.1.2 With correlation risk effect

We now study the elasticity between prudential regulation and bank income when there is a correlation risk effect. To do so, we consider the following ceiling rule in each market :

For market 1 :

$$\frac{A_1(Max)}{A_{1t}^{ceiling}(i)} = \frac{\gamma_{PA1}^{correl} e^{\gamma_{PA1}^{correl} \left(1 - \frac{A_{1t}(i)}{A_1(Max)}\right)}}{\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i)) \right)}$$

$$A_{1t}^{ceiling}(i) = \frac{\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i)) \right)}{\gamma_{PA1}^{correl} e^{\gamma_{PA1}^{correl} \left(1 - \frac{A_{1t}(i)}{A_1(Max)}\right)} (A_1(Max))^{-1}}$$

$$A_{1t}^{ceiling}(i) = \Gamma(A_{1t}(i); A_{2t}(i))$$

For market two :

$$\frac{A_2(Max)}{A_{2t}^{ceiling}(i)} = \frac{\gamma_{PA2}^{correl} e^{\gamma_{PA2}^{correl} \left(1 - \frac{A_{2t}(i)}{A_2(Max)}\right)}}{\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha_1 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{A_{1;2}(Max)}} \sqrt{(1 - \alpha_1) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{A_2(Max)}} \right) g(A_{2t}(i)) \right)}$$

$$A_{2t}^{ceiling}(i) = \frac{\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha_1 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{A_2(Max)}} \right) g(A_{2t}(i)) \right)}{\gamma_{PA2}^{correl} e^{\gamma_{PA2}^{correl} \left(1 - \frac{A_{2t}(i)}{A_{1;2}(Max)}\right)} (A_{1;2}(Max))^{-1}}$$

$$A_{2t}^{ceiling}(i) = \Psi(A_{1t}(i); A_{2t}(i))$$

Contrary to the previous part, there are interactions between the evolution of assets in both markets which leads to analyse four elasticity instead of two¹⁶ :

Elasticity between $A_1^{ceiling}$ and $A_1(i)$:

$$e_{A_1^{ceiling}/A_1(i)} = \Gamma'(A_{1t}(i); A_{2t}(i) | A_{1t}(i))$$

Elasticity between $A_1^{ceiling}$ and $A_2(i)$:

$$e_{A_1^{ceiling}/A_2(i)} = \Gamma'(A_{1t}(i); A_{2t}(i) | A_{2t}(i))$$

¹⁶Calculus of elasticities for two assets are detailed in the appendix.

Elasticity between $A_2^{ceiling}$ and $A_2(i)$:

$$e_{A_2^{ceiling}/A_2(i)} = \Psi(A_{1t}(i); A_{2t}(i) | A_{2t}(i))$$

Elasticity between $A_2^{ceiling}$ and $A_1(i)$:

$$e_{A_2^{ceiling}/A_1(i)} = \Psi(A_{1t}(i); A_{2t}(i) | A_{1t}(i))$$

Figure 11 (figure 12) depicts the impact of a 10% positive (negative) shock on bank asset size in market 1 on the evolution of four elasticities. The shock occurs at the fifteenth period and priors correspond to $\alpha = 1$ ¹⁷.

Notice that before the shock, elasticities of asset in market 1 and 2 (orange and yellow line) are lower than the ones studied without correlation risk effect. This is due to the PA tougher prudential policy in market 1 to limit the impact of its regulation on the other market. Furthermore, when the bank gets the same amount of asset in the two markets (such as the equilibrium situation), the authority is not incited to make a "cross-market" action (see mauve and green line).

We observe that an increase of bank size in market 1 leads the PA to absorb the shock partially because it wants to implement a progressive prudential policy (see orange line). Since there is a correlation, PA tries to manage the imbalance in both markets at the same time which leads the authority to coordinate its action between markets to dampen the effect of the shock.

4.2 Uncertainty priors and prudential policy

This part studies the impact of priors on prudential regulation and bank behaviour. When we have built variance, we have added two propositions to get a rational PA. The first one corresponds to the *proportional distribution of contributions* while the second one is the *proportional contribution in variance*. These two features lead to conclude that $\alpha_1 = \alpha_2 = \alpha$ for both markets.

Through this two propositions we have seen that the PA coordinates its actions on both markets to dampen shocks on bank size in a specific market. However, this coordination disappears when $\alpha = 0.5$ because it reflects the uncertainty of the PA regarding the contribution of two asset types on variances. To illustrates this idea, we consider the following variances :

¹⁷In the appendix, we write our equations and solutions according to α_1, α_2 to permit to readers to reproduce effects of simulation when proposition 2 does not hold.

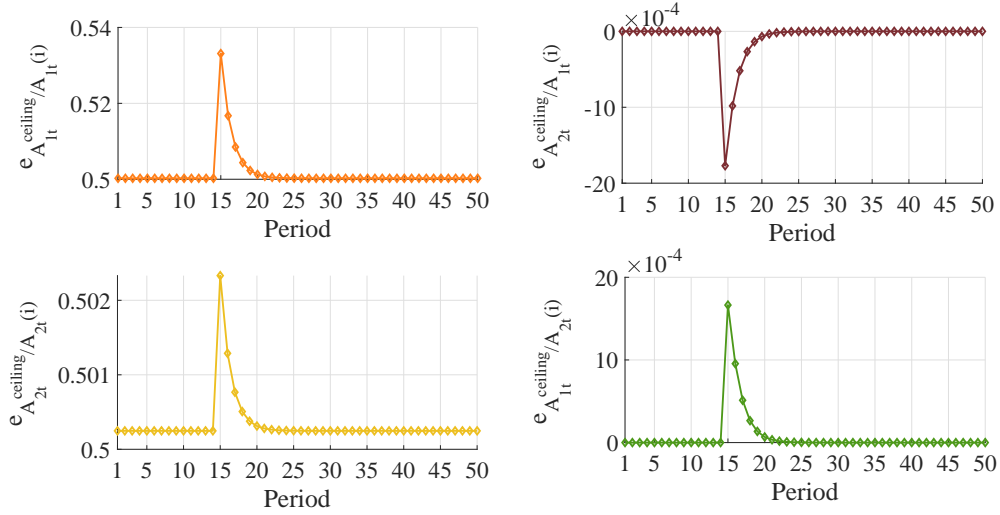


Figure 11: Impact of a 10% asset 1 size increase on the four elasticities.

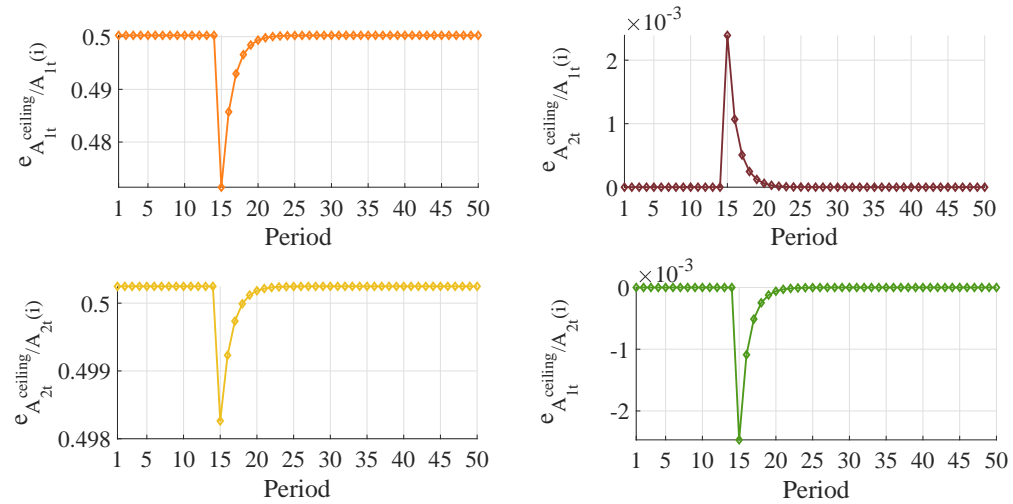


Figure 12: Impact of a 10% asset 1 size decrease on the four elasticities.

For market 1 :

$$Var(k_1) = \left(0.5 \frac{1}{k_1} + 0.5 \frac{1}{\theta k_2} \right)$$

$$Var(k_2) = \left(0.5 \frac{1}{k_1} + 0.5 \frac{1}{\theta k_2} \right)$$

For market 2 :

$$Var(k_1) = \left(0.5 \frac{\theta}{k_1} + 0.5 \frac{1}{k_2} \right)$$

$$Var(k_2) = \left(0.5 \frac{\theta}{k_1} + 0.5 \frac{1}{k_2} \right)$$

Then we draw on figure 13 the evolution of the four elasticities which also corresponds to the evolution of marginal income for each asset. We implement a positive shock on bank size in market 1 at the fifteenth period with prior equal to $\alpha = 0.5$. As we can see, market 1 PA is obliged to manage the shock on its market by reducing the marginal gain of bank progressively (orange line) but it doesn't react in the other market (green mauve and yellow line).

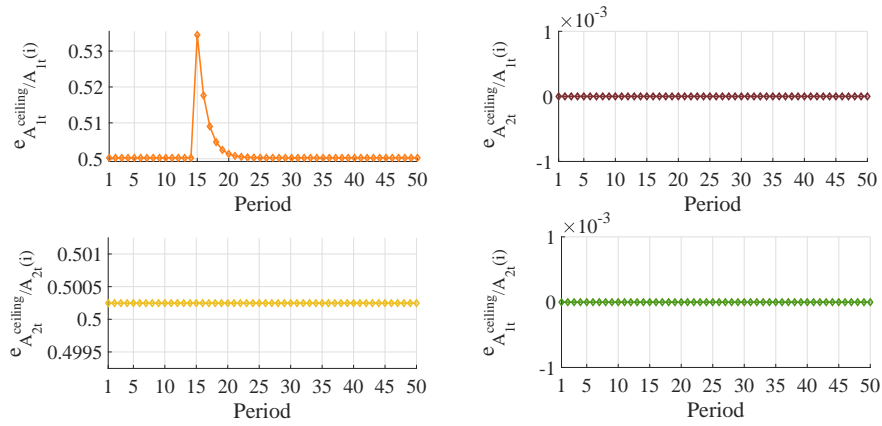


Figure 13: Impact of a asset 1 size positive shock on four elasticities with $\alpha = 0.5$.

5 Introduction of the mechanism in a DSGE model

This section is dedicated to see how introduce our mechanism in a Dynamic Stochastic General Equilibrium model (henceforth DSGE model). This type of model works with micro founded equations for each agent which composed the economy (households, firms, banks, authorities,...). These equations link these agents each other and lead to determine what is the intertemporal optimal choice for them. DSGE model can also be defined as a mathematical system composed of first order conditions of agents and constraints which drive the economy in a linear and/or non-linear fashion. The aim of these models is to show how interactions between agents evolves when exogenous stochastic shocks occur. The system is solved at each point in time and permits to draw

impulse responses function of key macroeconomic variables.

In this part, we will not set up a DSGE model but rather show how to introduce our mechanism inside a model already built. To do so, we consider the banking system from the Gerali *et al.* DSGE model (Gerali *et al.* (2010)). This system is composed of a two-stage bank when the first one corresponds to the wholesale branch which provides asset to the second type of bank, retailers. This structure allows to apply prudential regulation on wholesale banks to modify the credit supply offered by retailers to their customers. The author defines the behaviour of the wholesale banks by a profit function. These banks decide the quantity of loans they give to retailer and receive deposit from other agents. Its aim is to maximize the sum of its expected profit. If we suppose that the wholesale bank "i" provides two types of asset $A_{1t}(i)$ and $A_{2t}(i)$, then we can rewrite the programme of the Gerali *et al.* model by :

$$\begin{aligned} \max_{\{A_{1t}; A_{2t}; D_t\}} & R_t^{A_1} A_{1t}(i) + R_t^{A_2} A_{2t}(i) - R_t^D D_t - AC(A_{1t}; A_{2t}; BK_t) \\ \text{s.t} & A_{1t}(i) + A_{2t}(i) = D_t + BK_t \end{aligned}$$

Consequently :

$$\max_{\{A_{1t}; A_{2t}; D_t\}} R_t^{A_1} A_{1t}(i) + R_t^{A_2} A_{2t}(i) - R_t^D D_t - AC(A_{1t}; A_{2t}; BK_t)$$

Where $R_t^{A_1}$ and $R_t^{A_2}$ stand for the interest rate of the asset 1 and 2 respectively. R_t^D corresponds to the interest rate of deposit D_t whereas $A_{1t}(i)$ and $A_{2t}(i)$ represent level of bank "i" asset 1 and 2. Variable BK_t reflects bank capital. In the equation above, there is a quadratic adjustment cost $AC(A_{1t}; A_{2t}; BK_t)$ which depends of the gap between capital-to-asset ratio $\frac{BK_t}{A_{1t}+A_{2t}}$ and an optimal value v . This feature has to catch the capital requirement cost of bank when it departs from the optimal value. Since we would like to justify the existence of this cost in our mechanism, we take it out of the maximization programme. Thus, this latter can be rewritten as :

$$\max_{\{A_{1t}; A_{2t}; D_t\}} R_t^{A_1} A_{1t}(i) + R_t^{A_2} A_{2t}(i) - R_t^D D_t$$

5.1 Prudential constraint without correlation risk effect

To introduce the mechanism, we consider that the credit supply provided by the wholesale bank in time "t" corresponds to regulated level imposed by the PA. We suppose that the asset level required by the progressive prudential policy is achievable by the bank. As a result, the policy sequence is the following : in time "t", the authority implements a prudential policy which is applicable to banks immediately. Since banks is able to reach the new asset level, its credit supply (and its size) changes also in time "t".

Therefore, during this period, the bank wants to maximize its profit by anticipating its new asset level under prudential regulation :

$$\max_{\{A_{1t}; A_{2t}; D_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \Lambda_{0,t} \left\{ R_t^{A_1} \frac{1}{\gamma} e^{-\gamma \left(1 - \frac{A_{1,t}(i)}{A_1(Max)}\right)} A_1(Max) + R_t^{A_2} \frac{1}{\gamma} e^{-\gamma \left(1 - \frac{A_{2,t}(i)}{A_2(Max)}\right)} A_2(Max) - R_t^D D_t \right\}$$

The first order conditions for the two assets and deposit give the optimal interest rates fixed by the bank in each market :

$$R_t^{A_1*} = R_t^D e^{\gamma \left(1 - \frac{A_{1,t}(i)}{A_1(Max)}\right)} \quad (12)$$

$$R_t^{A_2*} = R_t^D e^{\gamma \left(1 - \frac{A_{2,t}(i)}{A_2(Max)}\right)} \quad (13)$$

Equation (12) and (13) shows that optimal rate for each asset type corresponds to the the deposit rate (riskless asset) adjusted by the inverse of the marginal prudential regulation effect on bank. Since a bank is regulated on its asset quantity, it would like to offset its loan quantity loss by increasing its loan price (interest rate). Consequently, at the optimum asset level, (i.e. when $A_{m,t}^*(i) = \frac{A_m(max)}{2}$ with $m = 1; 2$), the bank - which wants to reach $A_m(max)$ with an optimal interest rate $R_t^{A_1*} = R_t^D$ - would not be incited to resist to the prudential regulation if it had the possibility to double its interest rate to compensate its quantity loss. However, we have seen previously in figure 9 and 10 that the marginal effect of prudential regulation at the optimum is equal to 0.567, i.e, an inverse marginal effect of 1.7637 instead of 2 as the bank would have expected. However, the prudential regulation doesn't take into account the correlation effect that bank can use to offset losses by increasing asset quantity in the other market. This explains why the bank is ready to accept an optimal interest rate lower than the one expected. We are going to see that if the correlation risk effect is considered by the PA, then the bank will double its loans price.

Figure 14 (figure 15) depicts the dynamic path of the Optimal Interest Rate (henceforth OIR) when a positive (negative) shock hits the bank and increases (decreases) its size of 10%. When the shock occurs, the bank decides to reduce (increase) its OIR of almost 6% to manage its increase (reduction) of credit supply. The progressive and converging prudential regulation of the authority leads the bank to decrease (increase) step by step its OIR in order to limit its effect on the demand side.

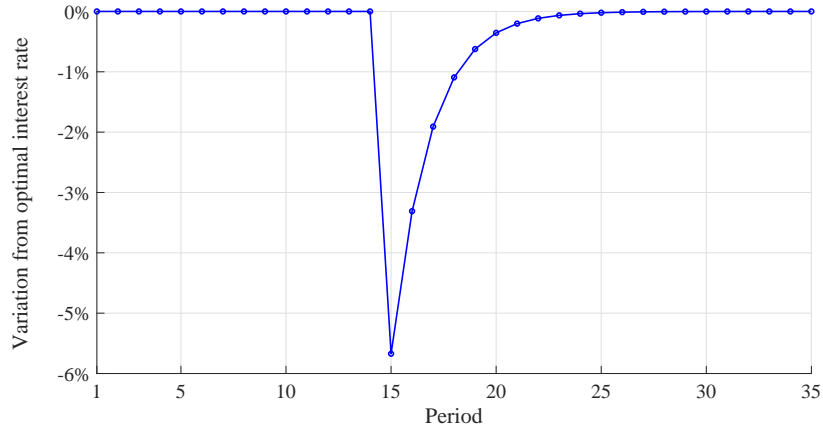


Figure 14: Impact of a 10% positive bank size shock on OIR.

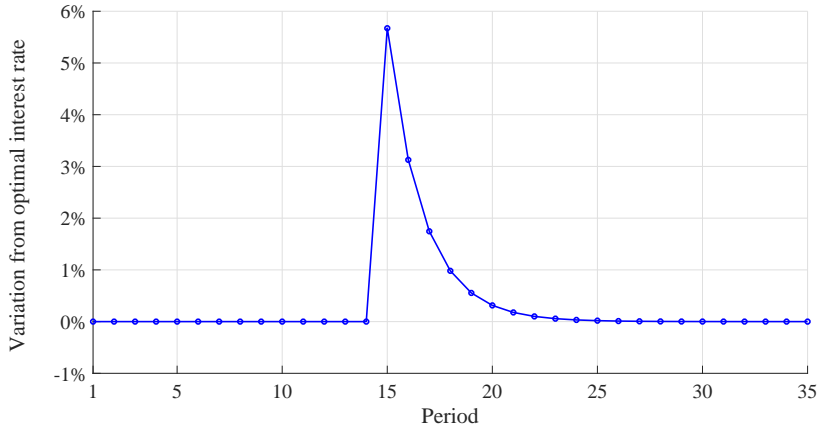


Figure 15: Impact of a 10% negative bank size shock on OIR.

5.2 Prudential constraint with correlation risk effect

When the correlation risk effect holds, the maximization programme of the wholesale bank profit function can be expressed as :

$$\max_{\{A_{1t}; A_{2t}; D_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \Lambda_{0,t} \{ R_t^{A_1} \Gamma(A_{1,t}(i); A_{2,t}(i)) + R_t^{A_2} \Psi(A_{1,t}(i); A_{2,t}(i)) - R_t^D D_t \}$$

The first order conditions for the two assets and deposit give the optimal interest rates fixed by the bank in each market :

$$R_t^{A_1^*} = \frac{R_t^D}{\Gamma(A_{1t}(i); A_{2t}(i)|A_{1t}(i))} \frac{1 - \frac{\Psi(A_{1t}(i); A_{2t}(i)|A_{1t}(i))}{\Psi(A_{1t}(i); A_{2t}(i)|A_{2t}(i))}}{\left(1 - \frac{\Gamma(A_{1t}(i); A_{2t}(i)|A_{2t}(i))\Psi(A_{1t}(i); A_{2t}(i)|A_{1t}(i))}{\Gamma(A_{1t}(i); A_{2t}(i)|A_{1t}(i))\Psi(A_{1t}(i); A_{2t}(i)|A_{2t}(i))}\right)} \quad (14)$$

$$R_t^{A_2^*} = \frac{R_t^D}{\Psi(A_{1t}(i); A_{2t}(i)|A_{2t}(i))} \frac{1 - \frac{\Gamma(A_{1t}(i); A_{2t}(i)|A_{2t}(i))}{\Gamma(A_{1t}(i); A_{2t}(i)|A_{1t}(i))}}{\left(1 - \frac{\Gamma(A_{1t}(i); A_{2t}(i)|A_{2t}(i))\Psi(A_{1t}(i); A_{2t}(i)|A_{1t}(i))}{\Gamma(A_{1t}(i); A_{2t}(i)|A_{1t}(i))\Psi(A_{1t}(i); A_{2t}(i)|A_{2t}(i))}\right)} \quad (15)$$

Equation (14) and (15) show interaction effects between both markets in the definition of the optimal interest rate in each of them. We have seen in the figure 11, 12 and 13 that the optimal asset level provides a null marginal prudential effect on cross-market when both markets get the same size ($\theta = 1$). Moreover, the action of the authority in each market corresponds to a marginal gain of 0.5002 unit instead of 1 as expected by the bank. This leads to get an optimal inverse marginal effect equals to 1.9992 which is very close to 2. This means that the authority prevents bank from optimizing its investment strategy through correlation channel. Hence, the later decides to set its optimal interest rate to offset the incurred losses by the prudential policy in its market.

This result confirms the soundness of our choice to construct a prudential rule whose the authority doesn't know the correlation between both markets. By considering the expected value of adjusted multiplier with correlation risk effect, the PA is able to constraint the bank to stay to the optimal level by allowing it to double its price. This allows the bank to compensate loans quantity losses to keep its profit unchanged¹⁸.

Figure 16 and 17 shows the impact of a 10% positive and negative bank size shock on the OIR in market 1. The graphs describe how correlation affects the OIR in market 2. Notice that a bank size increase of 10% leads to a OIR reduction of 6% in market 1 while it decreases of almost 1% the market 2 OIR. Since both loan markets get the same size¹⁹, the authority tries to incites bank to redirect the surplus of bank credit supply in market 1 toward market 2 by decreasing interest rate of asset 2. This allows to dampen the increase of the size bank in the former market and avoid to get a too powerful financial institution to be regulated. We also observe that positive and negative shock lead to

¹⁸This result holds when both markets have the same size, i.e., $\theta = 1$. When it is not the case, the inverse marginal effect is not close to 2 since bank knows that it has the opportunity to play with relative asset size in the correlation channel to compensate the loans quantity losses of prudential regulation in one market.

¹⁹When θ is different from 1, the asymmetrical size issue of both markets can lead the PA to manage the shock differently.

asymmetrical effect since the amplitude of interest reaction is higher when bank faces an increase of its size. This is due to the preference of bank to be closer to the critical size.

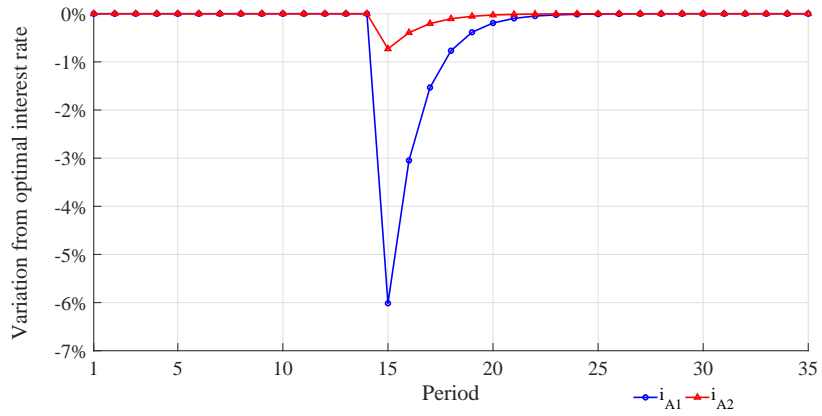


Figure 16: Impact of a 10% positive bank size shock on OIR.

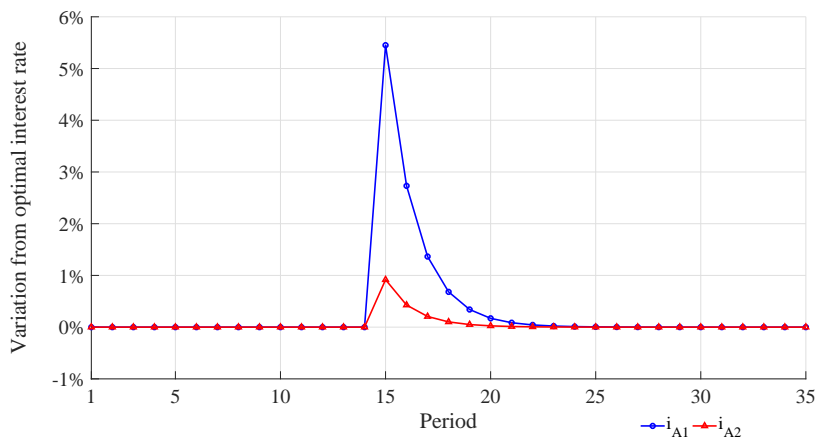


Figure 17: Impact of a 10% negative bank size shock on OIR.

6 Conclusion

We have developed a micro-founded model to give theoretical basis on the prudential mechanism. The setting up of a PA action anticipated by the bank to evaluate prudential policy leads to define an optimal asset size. This level corresponds to half the maximum critical value defined by the PA when assumptions on a "t-index" and a finite sum of risk profile hold. To confirm our suggestion about the temporal index, we have realized an empirical exercise based on a temporal index equals to a decade to represent the interval period between two prudential regulation (Basel rules). This exercise provides sounds results since the estimated relative asset size is not too far from the optimal one (i.e., $k^{opt} = 2$). The definition of an optimal asset level makes a converging prudential policy possible for real PA. This later may want to stabilize the credit sectors separately or simultaneously, by insuring a progressive convergence towards the optimal size. It decides to consider two scenarios in its prudential rule: with and without correlation between credit sectors. In absence of this effect, the PA only cares about the evolution of bank asset in each sector and the prudential constraint reduces the marginal income of the bank on its market. However, when the correlation is incorporated in the PA decision rule, this one is obliged to consider the stability of all credit sectors rather than to take into account only one. As we have seen, this additional issue incites the PA to implement a faster convergent policy. This is due to the coordinated actions of the PA on two different loans markets simultaneously oriented towards the stabilization of the market that has hit by a destabilizing shock. Finally, we have seen that this mechanism can be incorporated in a DSGE model with a two-stage banking system. A simple way is to integrate the mechanism into the optimal decision of wholesale banks which drive the credit supply of retailers. The constraint of prudential mechanism provides an other perspective of how the simultaneous consideration of different credit market segments in the definition of the prudential policy leads to manage bank sector through the optimal interest rate setting for each asset type.

Further research will be oriented towards the implementation of this mechanism in a large DSGE with different options for the bank' credit portfolio, allowing to take into consideration more realistic economic and financial environment.

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Appendix : Calculus of $\Gamma'(A_{1t}(i); A_{2t}(i))$ and $\Psi'(A_{1t}(i); A_{2t}(i))$

Calculus of $\Gamma'(A_{1t}(i); A_{2t}(i))$

We know that :

$$\Gamma(A_{1t}(i); A_{2t}(i)) = \frac{\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i)) \right)}{\gamma_{PA1}^{correl} e^{\gamma_{PA1}^{correl} \left(1 - \frac{A_{1t}(i)}{A_1(Max)}\right)} (A_1(Max))^{-1}}$$

Where :

$$g(A_{1t}(i)) = \frac{A_1(Max) - A_{1t}(i)}{A_{1t}(i) + A_{2t}(i)}$$

Futhermore :

$$\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i)) \right)$$

This is equivalent to express :

$$\begin{aligned} & \cosh \left(\left(\sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i)) \right) \\ & \times \prod_{j=1}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i)) \right) \end{aligned}$$

By using the hyperbolic sine property, we get :

$$\frac{\sinh \left(\left(2 \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i)) \right)}{\left(2 \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i))}$$

We rename the numerator by :

$$U = \sinh \left(\left(2 \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i)) \right)$$

And the denominator by :

$$V = \left(2 \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}} \right) g(A_{1t}(i))$$

Moreover we call Φ the ratio between U and V :

$$\Phi = \frac{U}{V}$$

And we express Δ as :

$$\Delta = e^{-\gamma_{PA1}^{correl} \left(1 - \frac{A_{1t}(i)}{A_1(Max)} \right)}$$

Derivative for $\Gamma(A_{1t}(i); A_{2t}(i)|A_{1t}(i))$

Since the gamma function is :

$$\Gamma(A_{1t}(i); A_{2t}(i)) = \Delta \times \Phi \times \frac{A_1(Max)}{\gamma_{PA1}^{correl}}$$

The derivative can be written as :

$$\Gamma'(A_{1t}(i); A_{2t}(i)|A_{1t}(i)) = [(\Delta|A_{1t}(i))'\Phi + (\Phi|A_{1t}(i))'\Delta] \times \frac{A_1(Max)}{\gamma_{PA1}^{correl}}$$

Where :

$$(\Delta|A_{1t}(i))' = \frac{\gamma_{PA1}^{correl}}{A_1(Max)} \Delta$$

And :

$$(\Phi|A_{1t}(i))' = \frac{((U|A_{1t}(i))' \times V) - ((V|A_{1t}(i))' \times U)}{V^2}$$

Futhermore

$$U = \sinh(2 \times Q \times P)$$

With:

$$Q = \sqrt{\alpha_1 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)}}$$

$$P = g(A_{1t}(i)) = \frac{A_1(Max) - A_{1t}(i)}{A_{1t}(i) + A_{2t}(i)}$$

Hence :

$$(U|A_{1t}(i))' = 2 [(Q|A_{1t}(i))'P + (P|A_{1t}(i))'Q] \cosh(2 \times Q \times P)$$

Where :

$$(P|A_{1t}(i))' = - \left(\frac{A_1(Max) + A_{2t}(i)}{(A_{1t}(i) + A_{2t}(i))^2} \right)$$

And :

$$(Q|A_{1t}(i))' = \frac{1}{2QA_1(Max)} \left[2\alpha_1(1 - \alpha_1) \frac{A_{1t}(i)}{A_1(Max)} + \alpha_1\alpha_2 \frac{A_{2t}(i)}{\theta A_2(Max)} + (1 - \alpha_1)(1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)} \right]$$

Moreover :

$$(V|A_{1t}(i))' = 2 [(Q|A_{1t}(i))'P + (P|A_{1t}(i))'Q]$$

Derivative for $\Gamma'(A_{1t}(i); A_{2t}(i)|A_{2t}(i))$

The derivative can be written as :

$$\Gamma'(A_{1t}(i); A_{2t}(i)|A_{2t}(i)) = \Delta \times (\Phi|A_{2t}(i))' \times \frac{A_1(Max)}{\gamma_{PA1}^{correl}}$$

Where :

$$(\Phi|A_{2t}(i))' = \frac{((U|A_{2t}(i))' \times V) - ((V|A_{2t}(i))' \times U)}{V^2}$$

Moreover :

$$(U|A_{2t}(i))' = 2 [(Q|A_{2t}(i))'P + (P|A_{2t}(i))'Q] \cosh(2 \times Q \times P)$$

With :

$$(P|A_{2t}(i))' = - \left(\frac{A_1(Max) - A_{1t}(i)}{(A_{1t}(i) + A_{2t}(i))^2} \right)$$

And :

$$(Q|A_{2t}(i))' = \frac{1}{2QA_2(Max)} \left[2\alpha_2(1 - \alpha_2) \frac{A_{2t}(i)}{\theta A_2(Max)} + \alpha_1\alpha_2 \frac{A_{1t}(i)}{A_1(Max)} + (1 - \alpha_1)(1 - \alpha_2) \frac{A_{1t}(i)}{A_1(Max)} \right]$$

Futhermore :

$$(V|A_{2t}(i))' = 2 [(Q|A_{2t}(i))'P + (P|A_{2t}(i))'Q]$$

Calculus of $\Psi'(A_{1t}(i); A_{2t}(i))$

We know that :

$$\Psi(A_{1t}(i); A_{2t}(i)) = \frac{\prod_{j=0}^{\infty} \cosh \left(\left(\frac{1}{2^j} \sqrt{\alpha_1 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{A_2(Max)}} \right) g(A_{2t}(i)) \right)}{\gamma_{PA2}^{correl} e^{\gamma_{PA2}^{correl} \left(1 - \frac{A_{2t}(i)}{A_2(Max)} \right)} (A_2(Max))^{-1}}$$

Where :

$$g(A_{2t}(i)) = \frac{A_2(Max) - A_{2t}(i)}{A_{1t}(i) + A_{2t}(i)}$$

By using the hyperbolic sine property, the cartesian product of hyperbolic cosine becomes :

$$\frac{\sinh \left(\left(2 \sqrt{\alpha_1 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{A_2(Max)}} \right) g(A_{2t}(i)) \right)}{\left(2 \sqrt{\alpha_1 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{A_2(Max)}} \right) g(A_{2t}(i))}$$

We rename the numerator by :

$$M = \sinh \left(\left(2 \sqrt{\alpha_1 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{A_2(Max)}} \right) g(A_{2t}(i)) \right)$$

And the denominator by :

$$N = \left(2 \sqrt{\alpha_1 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{A_2(Max)}} \right) g(A_{2t}(i))$$

Moreover we call Θ the ratio between U and V :

$$\Theta = \frac{M}{N}$$

And we express Λ as :

$$\Lambda = e^{-\gamma_{PA2}^{correl} \left(1 - \frac{A_{2t}(i)}{A_2(Max)} \right)}$$

Derivative for $\Psi(A_{1t}(i); A_{2t}(i)|A_{2t}(i))$

Since the psi function is :

$$\Psi(A_{1t}(i); A_{2t}(i)) = \Theta \times \Lambda \times \frac{A_2(Max)}{\gamma_{PA2}^{correl}}$$

The derivative can be written as :

$$\Psi'(A_{1t}(i); A_{2t}(i)|A_{2t}(i)) = [(\Theta|A_{2t}(i))'\Lambda + (\Lambda|A_{2t}(i))'\Theta] \times \frac{A_2(Max)}{\gamma_{PA2}^{correl}}$$

Where :

$$(\Lambda|A_{2t}(i))' = \frac{\gamma_{PA2}^{correl}}{A_2(Max)} \Lambda$$

And :

$$(\Theta|A_{2t}(i))' = \frac{((M|A_{2t}(i))' \times N) - ((N|A_{2t}(i))' \times M)}{N^2}$$

Futhermore

$$M = \sinh(2 \times S \times R)$$

With:

$$S = \sqrt{\alpha_1 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_2) \frac{A_{2t}(i)}{A_2(Max)}} \sqrt{(1 - \alpha_1) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_2 \frac{A_{2t}(i)}{A_2(Max)}}$$

$$R = g(A_{2t}(i)) = \frac{A_2(Max) - A_{2t}(i)}{A_{1t}(i) + A_{2t}(i)}$$

Hence :

$$(M|A_{2t}(i))' = 2 [(S|A_{2t}(i))'R + (R|A_{2t}(i))'S] \cosh(2 \times S \times R)$$

Where :

$$(R|A_{2t}(i))' = - \left(\frac{A_2(Max) + A_{1t}(i)}{(A_{1t}(i) + A_{2t}(i))^2} \right)$$

And :

$$(S|A_{1t}(i))' = \frac{1}{2SA_2(Max)} \left[2\alpha_1(1 - \alpha_1) \frac{A_{2t}(i)}{A_2(Max)} + \alpha_1\alpha_2 \frac{\theta A_{1t}(i)}{A_1(Max)} + (1 - \alpha_1)(1 - \alpha_2) \frac{\theta A_{1t}(i)}{A_1(Max)} \right]$$

Moreover :

$$(N|A_{2t}(i))' = 2 [(S|A_{2t}(i))'R + (R|A_{2t}(i))'S]$$

Derivative for $\Psi'(A_{1t}(i); A_{2t}(i)|A_{1t}(i))$

The derivative can be written as :

$$\Psi'(A_{1t}(i); A_{2t}(i)|A_{1t}(i)) = \Lambda \times (\Theta|A_{1t}(i))' \times \frac{A_2(Max)}{\gamma_{PA2}^{correl}}$$

Where :

$$(\Theta|A_{1t}(i))' = \frac{((M|A_{1t}(i))' \times N) - ((N|A_{1t}(i))' \times M)}{N^2}$$

Moreover :

$$(M|A_{1t}(i))' = 2 [(S|A_{1t}(i))'R + (R|A_{1t}(i))'S] \cosh(2 \times S \times R)$$

With :

$$(R|A_{1t}(i))' = - \left(\frac{A_2(Max) - A_{2t}(i)}{(A_{1t}(i) + A_{2t}(i))^2} \right)$$

And :

$$(S|A_{1t}(i))' = \frac{1}{2SA_2(Max)} \left[2\alpha_2(1 - \alpha_2) \frac{\theta A_{1t}(i)}{A_1(Max)} + \alpha_1\alpha_2 \frac{A_{2t}(i)}{A_2(Max)} + (1 - \alpha_1)(1 - \alpha_2) \frac{A_{2t}(i)}{A_2(Max)} \right]$$

Futhermore:

$$(N|A_{1t}(i))' = 2 [(S|A_{1t}(i))'R + (R|A_{1t}(i))'S]$$