Imperfect Credibility versus No Credibility of Optimal Monetary Policy.

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Abstract

When the probability of not reneging commitment of optimal monetary policy under quasi-commitment tends to zero, the limit of this equilibrium is qualitatively and quantitatively different from the discretion equilibrium assuming a zero probability of not reneging commitment for the classic example of the new-Keynesian Phillips curve. The impulse response functions and welfare are different. The policy rule parameter have opposite signs. The inflation auto-correlation parameter crosses a saddlenode bifurcation when shitfing to near-zero to zero probability of not reneging commitment. These results are obtained for all values of the elasticity of substitution between goods in monopolistic competition which enters in the welfare loss function and in the slope of the new-Keynesian Phillips curve.

JEL classification numbers: C61, C62, E31, E52, E58.

Keywords: Ramsey optimal policy under imperfact commitment, zero-credibility policy, Impulse response function, Welfare, New-Keynesian Phillips curve.

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1 Introduction

The degree of credibility of policy makers, measured by their probability of not reneging their commitment, is a key determinant of the efficiency of stabilization policy. This paper shows that when the probability of not reneging commitment of optimal monetary policy under "quasi-commitment" or "loose-commitment" (Schaumburg and Tambalotti (2007), Debortoli and Nunes (2014), Debortoli and Lakdawala (2016), Campbell and Weber (2018), Fujiwara, Kan and Sunakawa (2019)) tends to zero, the limit of this equilibrium is completely different from the discretion equilibrium assuming a zero probability of not reneging commitment for the usual example of the new-Keynesian Phillips curve.

The impulse response functions, welfare losses and initial anchors (or jump) of inflation are much larger with zero credibility than with near-zero credibility. The policy rule parameters have opposite signs, which causes a saddle-node bifurcation of the economy dynamic system with the inflation auto-correlation (or growth factor) parameter shifting from above one (zero credibility) to below one (near-zero credibility). Hence, a slight imperfect knowledge of structural parameters leads to inflation or deflation spirals for zero-credibility policy with a huge loss of welfare in the following two years. With the same slight imperfect knowledge of structural parameters, near-zero credibility, limited credibility and perfect credibility policy lean against inflation spirals with a moderate loss of welfare. The zero-credibility policy is a highly risky policy advice. It leans against inflation spirals only with an exact knowledge, with infinite precision, of the slope of the new-Keynesian Phillips curve and of other parameters of the monetary policy transmission mechanism.

These results are obtained for any value of the elasticity of substitution between goods larger than one. This parameter enters into the slope of the new-Keynesian Phillips curve and the welfare loss function. The key intuition is *with limited credibility, the policy maker's Lagrange multipliers of each private sector forward-looking variables are predetermined variables which are eliminated by assumption in the zero credibility model.* This originates a bifurcation of the economy dynamic system which is common to all dynamic stochastic general equilibrium (DSGE) model of the private sector solved with zero credibility for ever of the policy maker (optimal simple rule or discretionary policy) which have a lower number of stable eigenvalues than the same DSGE model of the private sector solved with Ramsey optimal policy with limited credibility.

Schaumburg and Tambalotti (2007, p.304) first statement that "*quasi-commitment converges to full commitment for the probability of reneging commitment tends to zero*" is valid. Their second statement that "*it also converges to discretion when the probability of reneging commitment tends to one*" is not valid, as demonstrated in this paper. This result is not marginal, because it suggests that there is no practical relevance for policy makers of the zero credibility discretion model, which is commonly used in macroeconomic theory. It lacks robustness to misspecification with respect to near-zero credibility and its representation of the lack of credibility for ever is extreme. By contrast, quasi-commitment models includes substitutes for extreme cases with near-zero credibility, such as a probability of not reneging commitment equal to 10^{-7}, which are robust to misspecification.

Section 2 presents Ramsey optimal policy under imperfect commitment and discretionary policy. Section 3 computes eigenvalues, policy rule parameters, initial anchors of inflation on the cost-push shock, impulse response functions, welfare and robustness to misspecification, in particular for the limit case of near-zero probability versus zero probability of not reneging commitment. The last section concludes.

2 Limited Credibility versus Zero Credibility For Ever

2.1 Ramsey optimal policy under quasi-commitment

In a monetary policy regime indexed by j, a policy maker may re-optimize on each future period with exogenous probability $1-q$ strictly below one ("quasi commitment" by Schaumburg and Tambalotti, 2007 and Debortoli and Nunes, 2014)). Following Schaumburg and Tambalotti (2007), we assume that the mandate to minimize the loss function is delegated to a sequence of policy makers with a commitment of random duration. The degree of credibility is modelled as if it is a change of policy-maker with a given probability of reneging commitment and re-optimizing optimal plans. The length of their tenure or "regime" depends on a sequence of exogenous i.i.d. Bernoulli signals $\{\eta_t\}_{t\geq 0}$ with $E_t\left[\eta_t\right]_{t\geq 0} = 1-q$, with $0 < q < 1$. If $\eta_t = 1$, a new policy maker takes office at the beginning of time t. Otherwise, the incumbent stays on. A higher probability q can be interpreted as a higher credibility. As seen below, this leads to use a "credibility adjusted" discount factor βq in the policy maker's optimal behavior.

Because structural parameters may change for a new regime k, long run equilibrium values may also change. Under regime j, policy plans solve the following problem, omitting subscript j for the central bank welfare preferences $\frac{\kappa}{\varepsilon}$ (see appendix 2), for the transmission mechanism parameter κ, the auto-correlation ρ of the cost-push shock u_t and its variance of its disturbances η_t:

$$V^{jk}\left(u_0\right) = -E_0 \sum_{t=0}^{t=+\infty} \left(\beta q\right)^t \left[\frac{1}{2}\left(\pi_t^2 + \frac{\kappa}{\varepsilon}x_t^2\right) + \beta\left(1-q\right)V^{jk}\left(u_t\right)\right] \tag{1}$$

$$\text{s.t. } \pi_t = \kappa x_t + \beta q E_t \pi_{t+1} + \beta\left(1-q\right)E_t\pi_{t+1}^k + u_t \text{ (Lagrange multiplier } \gamma_{t+1})$$

$$u_t = \rho u_{t-1} + \eta_{u,t}, \forall t \in \mathbb{N},\ u_0 \text{ given.}$$

where x_t represents the welfare-relevant output gap, i.e. the deviation between (log) output and its efficient level. π_t denotes the rate of inflation between periods $t-1$ and t. E_t denotes the expectation operator. The utility the central bank obtains is next period objectives change is denoted V^{jk}. Inflation expectations are an average between two terms in the new-Keynesian Phillips curve (appendix 1). The first term, with weight q is the inflation that would prevail under the current regime upon which there is commitment. The second term with weight $1-q$ is the inflation that would be implemented under the alternative regime, which is taken as given by the current central bank. The key change is that the narrow range of values for the discount factor around 0.99 for quarterly data (4% discount rate) is much wider for the "credibility weighted discount factor" of the policy maker: $\beta q \in \left]0, 0.99\right]$.

Differentiating the Lagrangian with respect to the policy instrument (output gap x_t)

and to the policy target (inflation π_t) yields the first order conditions:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \pi_t} = 0 : \pi_t + \gamma_{t+1} - \gamma_t = 0 \\ \frac{\partial \mathcal{L}}{\partial x_t} = 0 : \frac{\kappa}{\varepsilon} x_t - \kappa \gamma_{t+1} = 0 \end{cases} \Rightarrow \begin{cases} x_t = x_{t-1} - \varepsilon \pi_t \\ x_t = \varepsilon \gamma_{t+1} = \varepsilon (\gamma_t - \pi_t) \end{cases}$$

that must hold for $t = 1, 2, ...$ The central bank's Euler equation ($\frac{\partial \mathcal{L}}{\partial \pi_t} = 0$) links recursively the future or current value of central bank's policy instrument x_t to its current or past value x_{t-1}, because of the central bank's relative cost of changing her policy instrument is strictly positive $\alpha_x = \frac{\kappa}{\varepsilon} > 0$. This non-stationary Euler equation adds an unstable eigenvalue in the central bank's Hamiltonian system including three laws of motion of one forward-looking variable (inflation π_t) and of two predetermined variables (u_t, x_t) or (u_t, γ_t).

The *natural boundary condition* $\gamma_0 = 0$ minimizes the loss function with respect to inflation at the initial date:

$$\gamma_0 = 0 \Rightarrow x_{-1} = -\varepsilon \gamma_0 = 0 \text{ so that } \pi_0 = -\frac{1}{\varepsilon} x_0 \text{ or } x_0 = -\varepsilon \pi_0$$

It predetermines the policy instrument which allows to anchor the forward-looking policy target (inflation). The inflation Euler equation corresponding to period 0 is not an effective constraint for the central bank choosing its optimal plan in period 0. The former commitment to the value of the policy instrument of the previous period x_{-1} is not an effective constraint. The policy instrument is predetermined at the value zero $x_{-1} = 0$ at the period preceding the commitment. Combining the two first order conditions to eliminate the Lagrange multipliers yields the optimal initial anchor of forward inflation π_0 on the predetermined policy instrument x_0.

Ljungqvist and Sargent (2012, chapter 19) seek the stationary equilibrium process using the augmented discounted linear quadratic regulator solution of the Hamiltonian system (Hansen and Sargent (2007)) as an intermediate step (Chatelain and Ralf (2017) algorithm). This amount to seek a stable subspace of dimension two in a system of three equations including the marginal condition on the policy instrument (or on the Lagrange multiplier on inflation). The policy instrument is exactly correlated with private sectors variables:

$$x_t = F_\pi \pi_t + F_u u_t. \tag{2}$$

with solutions (see appendix) followed for $\beta = 0.99$:

$$F_\pi = \left(\frac{\lambda}{1-\lambda}\right)\varepsilon = 4.58 \text{ and } F_u = \frac{1}{\beta q \rho \lambda - 1} F_\pi = -1.51 F_\pi = -6.83$$
$$\lambda = \frac{1}{\beta q} - \frac{\kappa}{\beta q} F_\pi = \frac{1}{2}\left(1 + \frac{1}{\beta q} + \frac{\varepsilon\kappa}{\beta q}\right) - \sqrt{\frac{1}{4}\left(1 + \frac{1}{\beta q} + \frac{\varepsilon\kappa}{\beta q}\right)^2 - \frac{1}{\beta q}} = 0.43$$

We denote the inflation eigenvalue λ instead of δ in Gali (2015). It is the solution of the following characteristic polynomial:

$$\lambda^2 - \left(1 + \frac{1}{\beta q} + \frac{\kappa\varepsilon}{\beta q}\right)\lambda + \frac{1}{\beta q} = 0$$

The dynamics are unique with an initial optimal anchor of forward-looking inflation on the cost-push forcing variable, which is enforced by the optimal initial anchor of inflation on the policy instrument $\pi_0 = -\frac{1}{\varepsilon} x_0$. This optimal anchor rules out sunspot equilibria:

$$\pi_0 = \frac{\lambda}{1 - \beta q \rho \lambda} u_0 = \frac{-F_u}{F_\pi + \varepsilon} u_0 \text{ because}$$
$$x_0 = F_\pi \pi_0 + F_u u_0 \text{ and } \pi_0 = -\frac{1}{\varepsilon} x_0$$

The policy instrument (output gap), which can be substituted by the Lagrange multiplier of inflation, is optimally predetermined. The auto-regressive cost-push forcing variable is also predetermined. The optimal solution of the Hamiltonian system indeed satisfies Blanchard and Kahn (1980) determinacy condition with two stable eigenvalues: the inflation persistence parameter λ and the auto-regressive parameter ρ of the cost-push forcing variable.

2.2 Zero Credibility For Ever

With quasi-commitment, the probability of not reneging commitment could be infinitely small (near-zero credibility), but it remains strictly positive: for example, $q = 10^{-7} > 0$ with $q \in]0, 1]$, hence $\beta q \in]0, 0.99]$. An infinite horizon zero-credibility policy holds when the policy maker re-optimizes with certainty for all future periods: $q = 0$. This zero-credibility policy is mentioned as "discretionary policy". It is equivalent to the optimal simple rule in this model.

The central bank minimizes its loss function subject to the new-Keynesian Phillips curve *and* such that private sector and the central bank policy instrument reacts only to the contemporary *predetermined variable* u_t at all periods t with a *perfect correlation.* Each period the monetary authority is assumed to choose inflation and output gap in order to minimize the period losses

$$\pi_t^2 + \frac{\kappa}{\varepsilon} x_t^2 \tag{3}$$

subject to the constraint of the new-Keynesian Phillips curve where the expectation of future inflation is taken as given by the policy maker, because it is a function about future policy instruments (output gaps) and future cost-push shocks which cannot be currently influenced by the policy maker who has zero credibility for ever.

$$\pi_t = \kappa x_t + \beta E_t \left[\pi_{t+1} \right] + u_t \tag{4}$$

The optimality condition implies a policy rule with perfect negative correlation of the policy instrument (output gap) with the policy target (inflation) with constant parameter given by the opposite of the household's elasticity of substitution between goods:

$$x_t = -\varepsilon \pi_t \text{ for } t = 0, 1, 2, ... \text{ with } \varepsilon > 1. \tag{5}$$

Assuming that both the policy instrument and the policy target are forward-looking and that the cost-push shock is the only predetermined variable, Blanchard and Kahn

(1980) determinacy condition forces a unique solution which is given by the unique slope of the eigenvectors of the given stable eigenvalue $0 < \rho < 1$ of the cost-push shock:

$$\begin{pmatrix} \frac{1}{\beta} + \frac{\kappa}{\beta}\varepsilon & -\frac{1}{\beta} \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \rho \begin{pmatrix} \pi_t \\ u_t \end{pmatrix} \Rightarrow \left(\frac{1}{\beta} + \frac{\kappa}{\beta}\varepsilon - \rho \right) \pi_t = \frac{1}{\beta} u_t \tag{6}$$

There is an exact positive correlation between inflation and the cost-push shock:

$$\pi_t = \left(\frac{1}{1 - \beta\rho + \kappa\varepsilon} \right) u_t \tag{7}$$

Combining this equation with the policy rule leads to the exact negative correlation between output gap x_t and the cost-push shock u_t is:

$$x_t = -\varepsilon \left(\frac{1}{1 - \beta\rho + \kappa\varepsilon} \right) u_t \tag{8}$$

The policy maker lets the output gap and inflation deviate from their targets in exact proportion of the current value of the cost-push shock.

3 Bifurcation

3.1 Expected impulse response functions

The key equations are the expected impulse response functions, taking the expectations of random shocks (table 1). As detailed in the next sections, the policy parameter response F_π of the policy instrument to deviation of the policy target is positive for limited credibility and negative for zero-credibility. It is a bifurcation parameter which implies that the inflation eigenvalue is smaller than one for limited credibility and larger than one for zero-credibility. The initial jump of inflation (the first element of the column vector before u_0) have different formula. In the case of zero-credibility, the initial jump of inflation reduces the dimension of the dynamics to be of dimension one, whereas dynamics remains of dimension two for limited credibility. For this reason, there is no need for a second parameter F_u in the policy rule for zero-credibility, else it would not be identified. By contrast, a second parameter F_u in the policy rule is required in order to control the economy which evolves in two dimensions with limited credibility. The key result is *with limited credibility, the policy maker's Lagrange multipliers of each private sector forward-looking variables are predetermined variables which are eliminated by assumption in the zero credibility model.* Hence, determinacy implies a larger number of stable eigenvalues for limited credibility than in the zero-credibility case. This implies robustness to misspecification of the transmission mechanism for limited credibility and no robustness to misspecification for the zero-credibility model.

Table 1a: Expected impulse response functions.

Credibility	Impulse response functions following u_0	F_π	F_u
$q \in]0,1]$ Limited	$\begin{pmatrix}\pi_t \\ u_t\end{pmatrix} = \begin{pmatrix}\frac{1}{\beta q} - \frac{\kappa}{\beta q}F_\pi = \lambda & -\frac{1}{\beta q} - \frac{\kappa}{\beta q}F_u \\ 0 & \rho\end{pmatrix}^t \begin{pmatrix}\frac{\lambda}{1-\beta q\rho\lambda} \\ 1\end{pmatrix} u_0$	$\varepsilon\frac{\lambda}{1-\lambda}$	$\frac{-\lambda}{1-\beta q\rho\lambda}F_\pi$
$q = 0$ Zero	$\begin{pmatrix}\pi_t \\ u_t\end{pmatrix} = \begin{pmatrix}\frac{1}{\beta} + \frac{\kappa}{\beta}\varepsilon = \lambda_{ZC} & -\frac{1}{\beta} \\ 0 & \rho\end{pmatrix}^t \begin{pmatrix}\frac{1}{1-\beta\rho+\kappa\varepsilon} \\ 1\end{pmatrix} u_0$	$-\varepsilon$	0

Impulse response function are written in table 2 and represented on figure 1 for four different degrees of credibility q: 0 (zero credibility for ever), 10^{-7} (near-zero credibility), 0.5 (limited credibility), 1 (infinite horizon commitment). The calibration of parameters are taken from Gali (2015) with his corresponding impulse response functions for $q = 0$ and $q = 1$.

From table 2 and figure 1, with limited commitment, the parameters of the inflation dynamics (first row of the matrix and the jump vector) change marginally between $q = 1$ and $q = 10^{-7}$. Inflation eigenvalue increases from $\lambda = 0.43$ to 0.57. Inflation sensitivity with lagged cost-push shock shifts from -0.13 to -0.08. Inflation initial anchor on cost-push shock shifts from 0.65 to 0.57.

The shifts from near-zero credibility $q = 10^{-7}$ to zero credibility $q = 0$ are wide. Inflation eigenvalue increases from $\lambda = 0.57$ to 1.78 (multiplied by 3, crossing the bifurcation value 1). Inflation sensitivity with lagged cost-push shock shifts from -0.08 to -1.01 (multiplied by 12). Inflation initial anchor on cost-push shock shifts from 0.57 to 1.03 (multiplied by 1.8).

Table 2: Expected impulse response functions for $\rho = 0.8$, $\beta = 0.99$, $\varepsilon = 6$, $\kappa = 0.1275$ obtained with $\theta = 2/3$, $1-\alpha_L = 2/3$, $\sigma = 1$ and $\varphi = 1$ (see appendix 1).

Credibility	Impulse response functions following u_0	F_π	F_u
$q = 1$ commitment	$\begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} 0.43 & -0.13 \\ 0 & 0.8 \end{pmatrix}^t \begin{pmatrix} 0.65 \\ 1 \end{pmatrix} u_0$	4.51	−6.83
$q = 0.5$ quasi-commitment	$\begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} 0.50 & -0.12 \\ 0 & 0.8 \end{pmatrix}^t \begin{pmatrix} 0.62 \\ 1 \end{pmatrix} u_0$	5.92	−7.36
$q = 10^{-7}$ near zero commit.	$\begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} 0.57 & -0.08 \\ 0 & 0.8 \end{pmatrix}^t \begin{pmatrix} 0.57 \\ 1 \end{pmatrix} u_0$	7.84	−7.84
$q = 0$ discretion	$\begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} 1.78 & -1.01 \\ 0 & 0.8 \end{pmatrix}^t \begin{pmatrix} 1.03 \\ 1 \end{pmatrix} u_0$	−6	0

The impulse response function of inflation of zero credibility is markedly over the impulse response functions of inflation with limited credibility, including near-zero credibility. This is reflected in the evaluation of the relative welfare loss.

3.2 Welfare Losses

The expected loss function is for zero probability of not reneging commitment ($q = 0$) is given by:

$$W(q=0) = -\frac{1}{2}\sum_{t=0}^{t=+\infty} \beta^t \left(\pi_t^2 + \frac{\kappa}{\varepsilon}x_t^2\right) = -\frac{1}{2}\left(1+\frac{\kappa}{\varepsilon}\varepsilon^2\right)\left(\frac{1}{1+\kappa\varepsilon-\beta\rho}\right)^2 \sum_{t=0}^{t=+\infty} \beta^t \left(\rho^t u_0\right)^2$$

$$2W(q=0) = -\frac{1+\kappa\varepsilon}{(1+\kappa\varepsilon-\beta\rho)^2}\frac{u_0^2}{1-\beta\rho^2} = -5.09$$

We did not find a closed form formula for welfare losses in the case of limited credibility. We simulate the model over 200 periods in order to compute welfare for different elasticity and different credibility (table 3). For comparison with the welfare of infinite horizon regimes, the limited credibility welfare is arbitrarily computed using a discount factor of $\beta = 0.99$ instead of βq in order to take into account in a approximation the regimes which appears with probability $1-q$.

Table 3: Welfare loss in percentage of welfare loss with infinite horizon commitment ($w(q)=\frac{W(\beta q)}{W(\beta)}-1$, $\beta=0.99$, $q=1$) when varying ε and credibility q

-	-	$q=$	1	0.8	0.5	0.1	$\mathbf{10^{-7}}$	**0**
ε	$\kappa(\varepsilon)$	$\frac{\kappa(\varepsilon)}{\varepsilon}$	$2W(\beta)$	$w(q)$	$w(q)$	$w(q)$	$w(q)$	$w(q)$
3193	0.00032	10^{-7}	-2.119	2.8%	6.8%	10.8%	**2.1%**	**73%**
6	0.1275	0.02125	-2.688	3.2%	7.4%	10.9%	**0.03%**	**89%**
2.35	0.235	0.1	-3.489	3.6%	7.8%	10.2%	**8.6%**	**111%**
1	0.34	0.34	-7.971	4.1%	7.8%	7%	**23.6%**	**141%**

Because there is a wide gap between the large impulse response functions of zero-credibility $q=0$ with respect to near zero credibility $q=10^{-7}$, the welfare gap between near-zero versus zero credibility is also gigantic: from 71% if $\varepsilon=3193$ to 117% when ε tends to one.

When considering only limited credibility cases, the losses with respect to infinite horizon commitment are at most an increase of 24% of welfare losses in the limit case of the elasticity of substitution tending to 1, (corresponding to a large relative weight on output gap in the loss function of 0.34) for all the range of non-zero probabilities of reneging commitment.

3.3 Bifurcation of the inflation eigenvalue

This section demonstrates that shifting from limited credibility to zero credibility implies a *saddle-node bifurcation* of the dynamic system for the new-Keynesian Phillips curve transmission mechanism. The Lagrange multiplier on forward-looking inflation or the policy instrument is optimally predetermined for Ramsey optimal policy. The policy instrument is forward-looking with infinite horizon zero-credibility policy. This implies an additional stable eigenvalue for Ramsey optimal policy with respect to zero-credibility policy, according to Blanchard and Kahn (1980) determinacy condition.

Proposition 1 For any value of the elasticity of substitution between goods $\varepsilon>1$, there is a saddle-node bifurcation on the inflation eigenvalue when shifting from limited credibility $q \in]0,1]$ (stable eigenvalue $0<\lambda<1$) to zero credibility for ever $q=0$ (unstable eigenvalue $\lambda_{ZC}>1$).

Proof. For $\varepsilon \in]1,+\infty[$, we seek the limits of $\kappa\varepsilon$ which is an increasing function of ε.

$$
\begin{aligned}
\lim_{\varepsilon \to 1^+} \kappa\varepsilon &= \left(\sigma + \frac{\varphi+\alpha_L}{1-\alpha_L}\right)\frac{(1-\theta)(1-\beta\theta)}{\theta}(1-\alpha_L) = \kappa_{\max} \\
\lim_{\varepsilon \to +\infty} \kappa\varepsilon &= \left(\sigma + \frac{\varphi+\alpha_L}{1-\alpha_L}\right)\frac{(1-\theta)(1-\beta\theta)}{\theta}\frac{(1-\alpha_L)}{\alpha_L} = \frac{\kappa_{\max}}{\alpha_L} \text{ with } 0<\alpha_L<1 \\
&\Rightarrow \kappa_{\max} = 0.34 < \kappa\varepsilon < \frac{\kappa_{\max}}{\alpha_L} = 1.02
\end{aligned}
$$

Zero-credibility ($q=0$) inflation eigenvalue is an increasing function of $\kappa\varepsilon$. Its boundary conditions are:

$$
\begin{aligned}
&1 < \frac{1}{\beta} < \frac{1}{\beta} + \frac{1}{\beta}\kappa_{\max} < \lambda_{ZC} = \frac{1}{\beta} + \frac{1}{\beta}\kappa\varepsilon < \frac{1}{\beta} + \frac{1}{\beta}\frac{\kappa_{\max}}{\alpha_L} \\
&1 < 1.01 < 1.35 < \lambda_{ZC} = \frac{1}{\beta} + \frac{1}{\beta}\kappa\varepsilon < 2.04
\end{aligned}
$$

For limited credibility $q \in]0,1]$, λ is obtained solving a linear quadratic regulator model so that the inflation eigenvalue is necessarily within the range $[-1,1]$. However, the unit root case which is not necessarily excluded in the general linear quadratic regulator solution (Hansen and Sargent (2007)). More precisely, for the new-Keynesian Phillips curve transmission mechanism, limited credibility inflation eigenvalue is a decreasing function of $\kappa\varepsilon$, of ε, of βq and of q. To prove that their is a saddle-node bifurcation when shifting from limited credibility $q \in]0,1]$ (stable eigenvalue λ) to zero credibility for ever $q=0$ (unstable eigenvalue λ_{ZC}), it is sufficient to prove:

$$
\lim_{q \to 0^+}\lim_{\varepsilon \to 1^+} \lambda = \frac{1}{2}\left(1+\frac{1}{\beta q}+\frac{\kappa\varepsilon}{\beta q}\right) - \sqrt{\frac{1}{4}\left(1+\frac{1}{\beta q}+\frac{\kappa\varepsilon}{\beta q}\right)^2 - \frac{1}{\beta q}} = \frac{1}{1+\kappa_{\max}} < 1
$$

which is true because:

$$
\lim_{q \to 0^+} \frac{1}{2}\left(1+\frac{1}{\beta q}+\frac{1}{\beta q}\kappa_{\max}\right) - \sqrt{\frac{1}{4}\left(1+\frac{1}{\beta q}+\frac{1}{\beta q}\kappa_{\max}\right)^2 - \frac{1}{\beta q}} = \frac{1}{1+\kappa_{\max}} = 0.746 < 1
$$

and because when $q \to 0^+$:

$$\lambda \sim \frac{1+\kappa}{2\beta q}\left(1-\sqrt{1-\frac{4\beta q}{(1+\kappa)^2}}\right) \sim \frac{1+\kappa}{2\beta q}\frac{1}{2}\frac{4\beta q}{(1+\kappa)^2} = \frac{1}{1+\kappa}$$

Hence, there is a saddle-node bifurcation when shifting from limited credibility $q \in]0,1]$ with a stable inflation eigenvalue λ to zero credibility for ever $q = 0$ with an unstable inflation eigenvalue λ_{ZC}.

$$0 < \lambda_{\min} < \lambda < \frac{1}{1+\kappa_{\max}} < 1 < \frac{1+\kappa_{\max}}{\beta} < \lambda_{ZC} < \frac{1}{\beta} + \frac{1}{\beta}\frac{\kappa_{\max}}{\alpha_L}$$

QED.

Remark 2 *One also checks that there is no flip bifurcation within the regimes of limited credibility* $q \in]0,1]$*, seeking the lower bound of the inflation eigenvalue:*

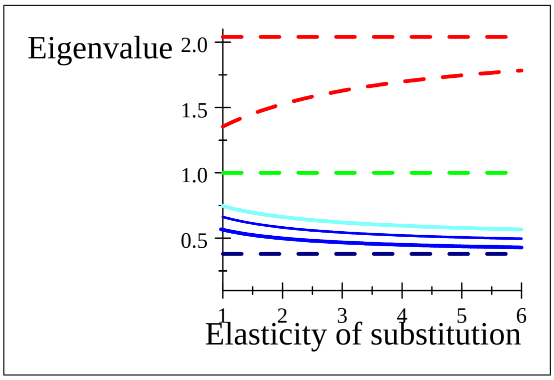
$$\lim_{q\to 1^-}\lim_{\varepsilon\to+\infty} \lambda = \frac{1}{2}\left(1+\frac{1}{\beta q}+\frac{\kappa\varepsilon}{\beta q}\right) - \sqrt{\frac{1}{4}\left(1+\frac{1}{\beta q}+\frac{\kappa\varepsilon}{\beta q}\right)^2 - \frac{1}{\beta q}} = \lambda_{\min} > -1$$

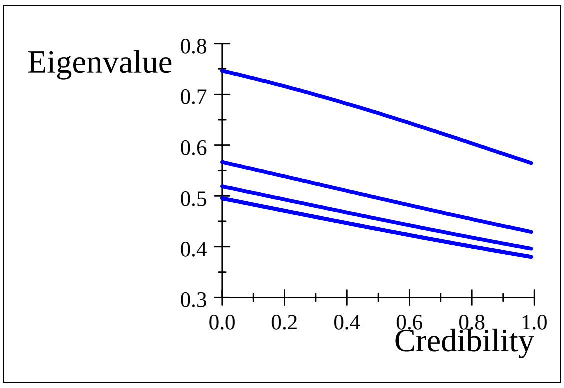
$$\lambda_{\min} = \frac{1}{2}\left(1+\frac{1}{\beta}+\frac{1}{\beta}\frac{\kappa_{\max}}{\alpha_L}\right) - \sqrt{\frac{1}{4}\left(1+\frac{1}{\beta}+\frac{1}{\beta}\frac{\kappa_{\max}}{\alpha_L}\right)^2 - \frac{1}{\beta}} = 0.379 > 0 > -1$$

■

Figure 2: Inflation eigenvalue $\frac{1}{\beta} - \frac{\kappa}{\beta}F_\pi$ as functions of the elasticity of substitution between differentiated goods for $\varepsilon \in [0,6]$ in the case where $q = 0$ (zero credibility for ever, dash line with its upper asymptote $\lambda_{\min}$) and in the case of limited credibility in four cases: $q = 0.001$ and $q = 10^{-7}$ (overlap on the top solid line below one), $q = 0.5$ (intermediate solid line), and finally $q = 1$ (bottom solid line, with a dash line below for its bottom asymptote). The dash line for 1 corresponds to the saddle-node bifurcation value separating discretion eigenvalue from eigenvalues with limited credibility.

Figure 3: Inflation eigenvalue as a decreasing function of credibility for $q \in]0,1]$ and of the elasticity of substitution between goods for different values: $\varepsilon = 1$ (top decreasing line), 6, 20 and finally 100 and 10^7 which overlap on the bottom decreasing line.





On figures 1 and 2, the limited credibility eigenvalue has an upper bound equal to $\frac{1}{1+\kappa_{\max}} = 0.746$ for near zero credibility q and near one elasticity of substitution between goods ε. The larger the credibility q, the lower the eigenvalue and the faster the convergence of inflation to equilibrium. The limit eigenvalues obtained with a near-zero probability of not reneging commitment $q = 10^{-7}$ are widely different from the eigenvalues obtained with a zero probability of reneging commitment $q = 0$ for all values of the elasticity of substitution larger than one.

3.4 Policy rule parameter as a bifurcation parameter

The feedback rule parameter F_π of the response of the policy instrument to deviations of inflation is a bifurcation parameter which drive the bifurcation of the inflation eigenvalue larger than one for zero credibility to smaller than one for limited credibility ($q \in]0,1]$). The inflation rule parameter is an affine and decreasing function of the inflation eigenvalue λ according to $\frac{1}{\kappa} - \frac{\beta q}{\kappa}\lambda$ and conversely.

Proposition 3 For any value of the elasticity of substitution between goods $\varepsilon > 1$, the inflation policy rule parameter F_π is positive for limited credibility. For zero-credibility, the inflation policy rule is negative and below -1.

Proof. One has:

$$-\infty < F_{\pi,ZC} = -\varepsilon < -1 < 0 \; < F_\pi = \frac{\lambda}{1-\lambda}\varepsilon = \frac{1}{\kappa} - \frac{\beta q}{\kappa}\lambda \tag{9}$$

For limited credibility:

$$-\infty < F_{\pi,ZC} = -\varepsilon < -1 < 0 \ < F_{\pi} \tag{10}$$

For limited credibility, the policy rule parameter of the response to inflation is a decreasing function of credibility q and an increasing function of the elasticity of substitution ε. To prove that the policy rule is positive, it is sufficient to prove:

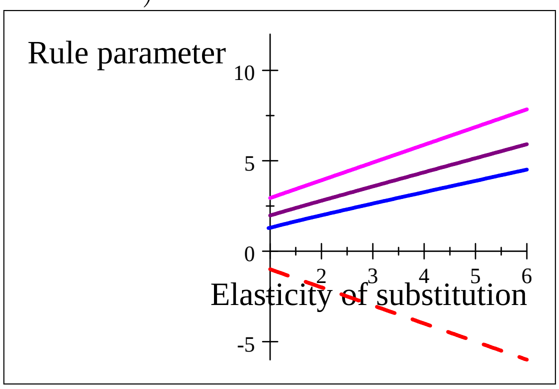
$$\lim_{q\to 1^-}\lim_{\varepsilon\to 1^+}\frac{1}{\kappa}-\frac{\beta q}{\kappa}\left(\frac{1}{2}\left(1+\frac{1}{\beta q}+\frac{\kappa\varepsilon}{\beta q}\right)-\sqrt{\frac{1}{4}\left(1+\frac{1}{\beta q}+\frac{\kappa\varepsilon}{\beta q}\right)^2-\frac{1}{\beta q}}\right)>0$$

When $q \to 1^-$ and when $\varepsilon \to 1^+$

$$F_{\pi}\sim\frac{1}{\kappa}-\frac{\beta}{\kappa}\left(\frac{1}{2}\left(1+\frac{1}{\beta}+\frac{\kappa}{\beta}\right)-\sqrt{\frac{1}{4}\left(1+\frac{1}{\beta}+\frac{\kappa}{\beta}\right)^2-\frac{1}{\beta}}\right)$$

In this case, one shows in the appendix that $F_{\pi} > 0$ is equivalent to $\kappa + \beta > \beta$ which is true because $\kappa \to \kappa_{\max} > 0$. ■

Figure 4. Policy rule parameters for different values of credibility q: 0 (dash line), 10^{-7} and 10^{-3} (overlap on the upper solid line), 0.5 (intermediate solid line), 1 (bottom solid line).



3.5 Initial anchor of inflation

Proposition 4 For any value of the elasticity of substitution between goods $\varepsilon > 1$, the initial anchor (or jump) of inflation on the cost-push shock is a decreasing function of the elasticity of substitution between goods for both limited credibility and zero credibility policy regimes. It is an increasing function of the limited credibility of the policy maker.

Proof. Output gap and inflation are exactly linearly related at the initial date $x_0 = -\varepsilon\pi_0$ for limited and zero-credibility case. The anchor of inflation on the cost-push shock are generally different between limited credibility versus zero credibility:

$$\pi_0 = \frac{\lambda}{1-\beta q\rho\lambda}u_0 \text{ versus } \pi_{0,ZC} = \frac{1}{1-\beta\rho+\kappa\varepsilon}u_0 \tag{11}$$

For zero credibility, the anchor of inflation is a decreasing function of $\kappa\varepsilon$ which is an increasing function of ε. As $\kappa_{\max} < \kappa\varepsilon < \frac{\kappa_{\max}}{\alpha_L}$, the zero credibility initial anchor of inflation (π_0/u_0) is bounded:

$$0.81 = \frac{1}{1-\beta\rho+\frac{\kappa_{\max}}{\alpha_L}} < \frac{1}{1-\beta\rho+\kappa\varepsilon} < \lim_{\varepsilon\to 1}\frac{1}{1-\beta\rho+\kappa\varepsilon} = \frac{1}{1-\beta\rho+\kappa_{\max}} = 1.82 \tag{12}$$

For limited credibility, the anchor of inflation is a decreasing function of $\kappa\varepsilon$ which is an increasing function of ε. As $\kappa_{\max} < \kappa\varepsilon < \frac{\kappa_{\max}}{\alpha_L}$, the zero credibility initial anchor of inflation (π_0/u_0) upper bound.

$$\lim_{q\to 1^-}\lim_{\varepsilon\to 1}\frac{\lambda}{1-\beta q\rho\lambda} = \lim_{q\to 1^-}\lim_{\varepsilon\to 1}\frac{\lambda}{1-\beta\rho\lambda} = 1.02$$

With:

$$\lim_{q\to 1^-}\lim_{\varepsilon\to 1}\lambda = \frac{1}{2}\left(1+\frac{1}{\beta}+\frac{\kappa_{\max}}{\beta}\right) - \sqrt{\frac{1}{4}\left(1+\frac{1}{\beta}+\frac{\kappa_{\max}}{\beta}\right)^2 - \frac{1}{\beta}} = 0.56$$

The initial anchor of near-zero credibility is always strictly smaller than the initial anchor in the case of zero credibility. The gap tends to zero when the auto-correlation of the forcing variable tends to zero and when the elasticity of substitution tends to one: $\rho \to 0$ and $\varepsilon \to 1$.

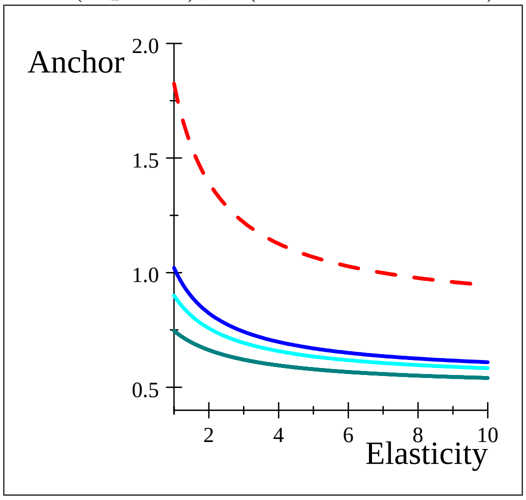
$$\lim_{q\to 0^+}\frac{\lambda}{1-\beta q\rho\lambda} = \lim_{q\to 0^+}\lambda \sim \frac{1}{1+\kappa} < \frac{1}{1-\beta\rho+\kappa\varepsilon}$$

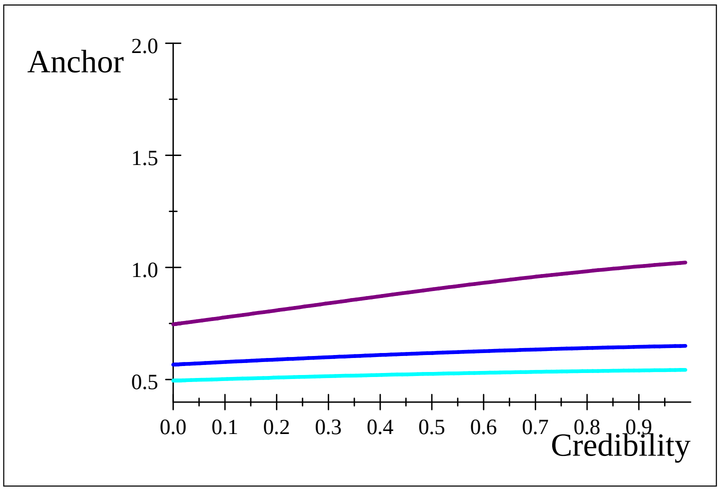
QED. ■

With $\rho = 0.8$, for any elasticity of substitution and for any probability of not reneging commitment, the zero credibility initial anchor of inflation is much higher (+80%) than the limited credibility initial anchor of inflation (figure 5).

Figure 5: Initial anchor of inflation as a decreasing function of the elasticity of substitution for $q = 0$ (dash line top), $q = 1$ (solid line, second line from top), $q = 0.5$ (solid line, third curve from top), $q = 10^{-7}$ with a value equal to the inflation eigenvalue λ (solid line, bottom curve) .

Figure 6: Initial anchor of inflation as an increasing function of credibility $q \in]0, 1]$ for $\varepsilon = 1$ (top line), 6 (intermediate line) and 10^7 (bottom line).





As seen in figure 6, there is a potential trade-off within the cases of limited credibility: more credibility (a higher $q \in]0, 1]$) implies faster convergence on subsequent period over a longer expected duration measured by the inflation eigenvalue, but it allows a higher initial anchor of inflation which slows convergence.

3.6 Robustness to misspecification

We assume that there is a misspecification by the private sector and the policy maker on their exact knowledge of parameters $\beta, \rho, \kappa, \varepsilon, u_0$ so that the initial anchor of inflation π_0 deviates from $\pm 10\%$ with respect to its value with exact knowledge of parameters. This assumption is grounded by a number of major measurement issues:

1. Inflation π_0 is not measured with exact precision. This error is related to consumer price index versus core inflation, quality adjusted bias and the revisions of national

accounts.

2. A major source of new-Keynesian uncertainty is the measurement of the unobservable cost-push shock initial value u_0 depending on its past value u_{-1}, on its auto-correlation ρ and on the disturbance η_0. The cost-push shock is indirectly measured an auto-correlated residual. It faces identification issues when an additional lag is included for inflation in hybrid Phillips curve. As a residual, it varies widely depending on misspecfication of inflation dynamics.

3. The estimated slope $\kappa\left(\beta, \varepsilon, \alpha_L, \theta, \sigma, \varphi\right)$ of the new-Keynesian Phillips curve in only known with a standard error. It sign is even uncertain (Mavroeidis et al. (2015)). It is itself a function of six not so precisely known structural parameters $\kappa\left(\beta, \varepsilon, \eta, \theta, \sigma, \varphi\right)$, in particular the proportion of firms θ who do not reset their price at each quarter.

4. The elasticity of substitution between differentiated inputs ε in monopolistic competition which enters into welfare relative weight is not precisely known. Some authors may refer to the measurement of Lerner index which are themselves lacking precision, with a calibration of $\varepsilon = 11$ instead of Gali (2015) calibration of $\varepsilon = 6$.

5. The policy maker discount factor β may vary much more with a adjusted discount factor βq depending on the probability q of not reoptimizing. For example, Debortoli and Lakdawala (2016) point estimate is $\widehat{q} = 0.81$ in a 95% confidence interval $[0.777, 0.851]$.

We compute two impulse response functions of out of equilibrium path when facing $\pm 10\%$ error on the initial anchor of inflation.

For near zero credibility ($q = 10^{-7}$), the error gap of 10% with respect to the perfect knowledge optimal path at the initial date is reduced to less than 1% after eight quarters (figure 8).

For zero credibility ($q = 0$), the error gap of 10% with respect to the perfect knowledge optimal path at the initial date is increased to 110% after four quarters and to 270%

after eight quarters (figure 9) with inflation or deflation spirals. After six quarters, the divergence of inflation reaches $+1\%$ additional inflation with $+10\%$ error or -2% additional deflation with -10% error with respect to the perfect knowledge impulse response function.

In the perfect knowledge case, which has a probability zero for practitioners of stabilization policy, the expected impulse response function may suggest that discretionary policy leans against inflation spirals, while using inflation rule parameters destabilizing the inflation eigenvalue. By contrast, in the imperfect knowledge case with zero credibility, the outcome of discretionary policy is a probability equal to one of inflation or deflation spirals. The core behavioral hypothesis that a policy maker sticks to an *exactly zero* probability of not reneging commitment *for ever* is also an assumption with a probability zero for practitioners of stabilization policy.

3.7 Conclusion

Even in the most favorable case of an elasticity of substitution between goods tending to one, the limited-credibility equilibrium when the probability to renege commitment tends to zero is *never* the limit of the zero credibility for ever equilibrium: positive sign versus negative sign of the response of the policy instrument to inflation, stability versus instability of the inflation eigenvalue, small versus large initial anchor of inflation, small versus large magnitude of welfare loss, robustness versus lack of robustness to a large range of misspecification and measurement error.

The bifurcation between zero credibility versus limited credibility is a general result for any linear model of the private sector with any number of lags and any number of variables including at least one forward-looking variable with a policy maker quadratic loss function. With limited credibility, the policy maker's Lagrange multipliers of each private sector forward-looking variables are predetermined variables which are eliminated by assumption in the zero credibility model (e.g. Chatelain and Ralf (2017a)). Hence, Blanchard and Kahn (1980) determinacy condition implies more stable eigenvalues with limited credibility model with respect to zero credibility.

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3.8 Appendix 1: New-Keynesian Phillips Curve

The reference new-Keynesian Phillips curve is the monetary policy transmission mechanism:

$$\pi_t = \beta E_t\left[\pi_{t+1}\right] + \kappa x_t + u_t \text{ where } \kappa > 0,\ 0 < \beta < 1 \tag{13}$$

where x_t represents the welfare-relevant output gap, i.e. the deviation between (log) output and its efficient level. π_t denotes the rate of inflation between periods $t-1$ and t. u_t denotes a cost-push shock. β denotes the discount factor. E_t denotes the expectation operator. The cost push shock u_t includes an exogenous auto-regressive component:

$$u_t = \rho u_{t-1} + \eta_{u,t} \text{ where } 0 < \rho < 1 \text{ and } \eta_{u,t} \text{ i.i.d. normal } N\left(0, \sigma_u^2\right) \tag{14}$$

The disturbances $\eta_{u,t}$ are identically and independently distributed (i.i.d.) according to a normal distribution with constant variance σ_u^2.

The reduced-form parameter (denoted κ) of the slope of the new-Keynesian Phillips curve relates inflation to marginal cost or to the output gap. It is a non-linear function of four preferences and two technology parameters:

$$\lim_{\varepsilon \to +\infty} \kappa = 0 < \kappa = \left(\sigma + \frac{\varphi + \alpha_L}{1-\alpha_L}\right) \frac{(1-\theta)(1-\beta\theta)}{\theta} \frac{(1-\alpha_L)}{(1-\alpha_L+\alpha_L\varepsilon)} < \kappa_{\max} = \lim_{\varepsilon \to 1^+} \kappa \tag{15}$$

with $\varepsilon > 1$, $0 < \beta, \alpha_L, \theta < 1$, $\sigma > 0$, $\varphi > 0$.

$$\kappa_{\max} = \lim_{\varepsilon \to 1^+} \kappa = \left(\sigma + \frac{\varphi + \alpha_L}{1-\alpha_L}\right) \frac{(1-\theta)(1-\beta\theta)}{\theta} (1-\alpha_L)$$

Gali's (2015, chapter 3) calibration of structural parameter is as follows. The auto-correlation of the cost-push shock is $\rho = 0.8$. The representative household discount factor $\beta = 0.99$. It is also assumed $\sigma = 1$ (log utility) and $\varphi = 1$ (a unitary Frisch elasticity of labor supply). For the production functions of the firms, the measure of decreasing returns to scale of labor is $0 < \alpha_L = 1/3 < 1$ (the production function is $Y = A_t L^{1-\alpha_L}$

where Y is output, L is labor, A_t represents the level of technology and $1-\alpha_L=2/3$). The proportion of firms who do not reset their price each period $0<\theta=2/3<1$ which corresponds to an average price duration of three quarters. The household's elasticity of substitution between each differentiated intermediate goods is $\varepsilon=6>1$, which is assumed to be larger than one. The maximal value of the slope of the new-Keynesian Phillips curve when varying the elasticity of substitution between intermediate goods, $\kappa_{\max}=0.34$ is obtained when the elasticity of substitution tends to one. In what follows, we provide Gali (2015) numerical values besides general solution in order to keep insights on orders of magnitude.

$$\kappa=\left(1+\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)\frac{\left(1-\frac{2}{3}\right)\left(1-0.99\frac{2}{3}\right)}{\frac{2}{3}}\frac{\left(1-\frac{1}{3}\right)}{\left(1-\frac{1}{3}+\frac{1}{3}\varepsilon\right)}=\frac{1.02}{2+\varepsilon}$$
$$\kappa\left(\varepsilon=6\right)=0.1275<\kappa_{\max}=0.34.$$

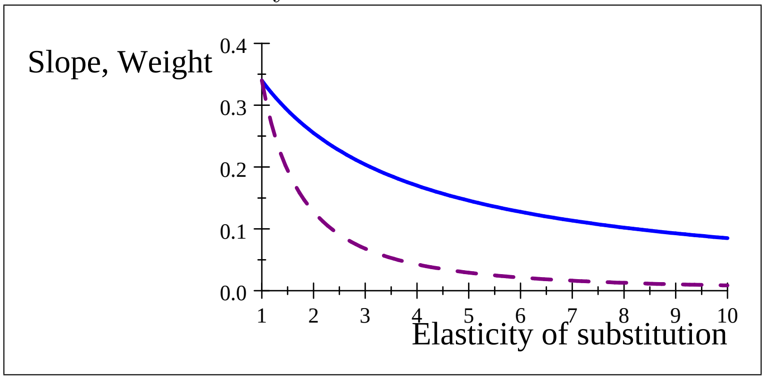
3.9 Appendix 2: Welfare loss function

In a monetary policy regime indexed by j, a policy maker has a period loss function $\frac{1}{2}\left(\pi_t^2+\alpha_{x,j}x_t^2\right)$. If the policy maker is maximizing welfare, its preferences α_x depend on structural parameters of the transmission mechanism (Gali (2015):

$$0<\alpha_x=\frac{\kappa}{\varepsilon}=\left(\sigma+\frac{\varphi+\alpha_L}{1-\alpha_L}\right)\frac{(1-\theta)(1-\beta\theta)}{\theta}\frac{(1-\alpha_L)}{(1-\alpha_L+\alpha_L\varepsilon)}\frac{1}{\varepsilon}<\kappa<\kappa_{max}$$
$$\alpha_x=\frac{1.02}{2+\varepsilon}\frac{1}{\varepsilon}=0.02125<\ \text{ if } \varepsilon=6$$

With Gali's (2015) calibration of structural parameters: $\kappa=\frac{1.02}{2+\varepsilon}$ and $\varepsilon=6$, the relative weight of the variance of the policy instrument (output gap) is a very low proportion (2.125%) of the weight on the variance of the policy target (inflation). This is a very low relative cost of changing the policy instrument which implies a fast convergence of the policy target. Both the slope κ of the monetary transmission mechanism and the policy maker's preferences are *decreasing* functions of the household's elasticity of substitution between each differentiated goods, whose values varies in $\varepsilon \in]1, +\infty[$ (figures 1 and 2).

Figure 1: Slope of the new-Keynesian Phillips curve (solid curve, $\kappa = \frac{1.02}{2+\varepsilon}$) and relative welfare cost of changing the policy instrument (dash curve, $\alpha_x = \frac{1.02}{2+\varepsilon}\varepsilon$) as a function of the elasticity of substitution between differentiated goods.



When the elasticity of substitution tends to one, the slope of the new Keynesian Phillips curve is equal to the relative welfare weight on output gap in proportion of the weight on the variance of the policy target (inflation) in the loss function at its maximal value ($\kappa_{\max} = 0.34 = \alpha_{x,\max}$ with Gali's calibration).

When the elasticity of substitution of differentiated goods tends to infinity (all other parameters being unchanged), the convergence to zero of the relative welfare weight of the policy instrument in the loss function is faster than the one of the slope of the new-Keynesian Phillips curve.

3.10 Appendix 3: Augmented Discounted Linear Quadratic Regulator

The new-Keynesian Phillips curve can be written as a function of the Lagrange multiplier where $\kappa > 0$, $0 < \beta < 1$ and $0 < q < 1$ (Debortoli and Nunes (2014, appendix A). We keep Gali (2015) chapter 5 γ_{t+1} notation of the Lagrange multiplier with one step ahead subscript: it corresponds to Debortoli and Nunes (2014) notation λ_t. Our notation for the stable eigenvalue λ corresponds to Debortoli and Nunes (2014) notations "$\psi_y = 1/\gamma$".

$$E_t \pi_{t+1} + \frac{\kappa\varepsilon}{\beta q}\gamma_{t+1} = \frac{1}{\beta q}\pi_t - \frac{1}{\beta q}u_t - \frac{1-q}{q}E_t \pi^j_{t+1}$$

The solution of the Hamiltonian system are based on the demonstrations of the augmented discounted linear quadratic regulator in Anderson, Hansen, McGrattan and Sargent [1996], following the steps in Chatelain and Ralf (2017a):

$$\begin{pmatrix} 1 & \frac{\kappa\varepsilon}{\beta q} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ \gamma_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta q} & 0 & \frac{-1}{\beta q} \\ -1 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ \gamma_t \\ u_t \end{pmatrix} + \begin{pmatrix} -\frac{1-q}{q} E_t \pi_{t+1}^j \\ 0 \\ 0 \end{pmatrix}$$

The Hamiltonian system is:

$$\begin{pmatrix} \pi_{t+1} \\ \gamma_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta q} + \frac{\kappa\varepsilon}{\beta q} & -\frac{\kappa\varepsilon}{\beta q} & -\frac{1}{\beta q} \\ -1 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ \gamma_t \\ u_t \end{pmatrix} + \begin{pmatrix} -\frac{1-q}{q} E_t \pi_{t+1}^j \\ 0 \\ 0 \end{pmatrix}$$

The characteristic polynomial of this upper square matrix is:

$$\lambda^2 - \left(1 + \frac{1}{\beta q} + \frac{\kappa\varepsilon}{\beta q}\right)\lambda + \frac{1}{\beta q} = 0$$

The Hamiltonian matrix has two stable roots ρ and λ (λ is denoted δ in Gali (2015)) and one unstable root $\frac{1}{\beta q \lambda}$. The determinant of the matrix is $\rho\lambda\frac{1}{\beta q\lambda} = \rho\frac{1}{\beta q}$. Then $\lambda < \sqrt{\frac{1}{\beta q}} < \frac{1}{\beta q \lambda}$. The trace of the matrix is

$$\lambda = \frac{1}{2}\left(1 + \frac{1}{\beta q} + \frac{\kappa\varepsilon}{\beta q} - \sqrt{\left(1 + \frac{1}{\beta q} + \frac{\kappa\varepsilon}{\beta q}\right)^2 - \frac{4}{\beta q}}\right)$$

Policy rule parameter function of $\lambda(\varepsilon)$ and ε:

$$(1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right) = -\frac{\kappa\varepsilon}{\beta q} \Longrightarrow \left(\frac{1-\lambda}{\beta q \lambda}\right)\left(\frac{\beta q \lambda - 1}{\kappa}\right) = -\frac{\varepsilon}{\beta q} \Longrightarrow$$
$$F_\pi = \frac{1 - \beta q \lambda}{\kappa} = \left(\frac{\lambda}{1-\lambda}\right)\varepsilon$$

Hamiltonian system function of the stable eigenvalue λ (eliminating ε):

$$\begin{pmatrix} \pi_{t+1} \\ \gamma_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda + \frac{1}{\beta q \lambda} - 1 & 1 + \frac{1}{\beta q} - \lambda - \frac{1}{\beta q \lambda} & -\frac{1}{\beta q} \\ -1 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_{t+1} \\ \gamma_{t+1} \\ u_{t+1} \end{pmatrix}$$

Proposition A1: Solution of Ricatti and Sylvester equation: Rule parameters P_u and P_z of the response of the Lagrange multiplier on inflation to exogenous variables:

$$\gamma_t = P_\pi \pi_t + P_u u_t \tag{16}$$

$$P_\pi = \frac{1}{1-\lambda} > 0,\ P_u = \frac{1}{1-\lambda} \frac{\frac{1}{\beta q}}{\rho - \frac{1}{\beta q \lambda}} = \frac{1}{1-\lambda} \frac{\lambda}{\beta q \lambda \rho - 1} < 0 \tag{17}$$

Demonstration: We use the method of undetermined coefficients of Anderson, Hansen, McGrattan and Sargent's (1996), section 5. The solution is the one that stabilizes the state-costate vector for any initialization of inflation π_0 and of the exogenous variables u_0 in a stable subspace of dimension two within a space of dimension three (π_t, γ_t, u_t) of the Hamiltonian system. We seek a characterization of the Lagrange multiplier γ_t of the form:

$$\gamma_t = P_\pi \pi_t + P_u u_t.$$

To deduce the control law associated with vector (P_π, P_u), we substitute it into the Hamiltonian system:

$$\begin{pmatrix} \pi_{t+1} \\ P_\pi \pi_{t+1} + P_u u_{t+1} \\ u_{t+1} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\beta q} - (1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right) & (1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right) & -\frac{1}{\beta q} \\ -1 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \pi_t \\ P_\pi \pi_t + P_u u_t \\ u_t \end{pmatrix}$$

We write the last two equations in this system separately:

$$P_\pi \pi_{t+1} + P_u u_{t+1} = (P_\pi - 1)\,\pi_t + P_u u_t$$
$$u_{t+1} = \rho u_t$$

It follows that:

$$\pi_{t+1} = \frac{P_\pi - 1}{P_\pi}\pi_t + \frac{(1-\rho)\,P_u}{P_\pi}u_t$$

The first equation is such that:

$$\pi_{t+1} = \left[\frac{1}{\beta q} - (1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right)\right]\pi_t + (1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right)(P_\pi \pi_t + P_u u_t) - \frac{1}{\beta q}u_t$$

Factorizing:

$$\pi_{t+1} = \left[\frac{1}{\beta q} - (1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right) + (1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right)P_\pi\right]\pi_t + \left[(1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right)P_u - \frac{1}{\beta q}\right]u_t$$

The method of undetermined coefficients implies for the first term:

$$\frac{P_\pi - 1}{P_\pi} = \frac{1}{\beta q} + (1-\lambda)\left(1 - \frac{1}{\beta q \lambda}\right)(P_\pi - 1)$$
$$P_\pi = \frac{1}{1-\lambda}$$

For the second term:

$$\frac{(1-\rho)\,P_u}{P_\pi} = (1-\lambda)\left(1-\frac{1}{\beta q\lambda}\right)P_u - \frac{1}{\beta q} \Rightarrow$$
$$\frac{1}{\beta q} = \left(1-\frac{1}{\beta q\lambda} - 1 + \rho\right)(1-\lambda)\,P_u \Rightarrow$$
$$P_u = \frac{1}{1-\lambda}\frac{\frac{1}{\beta q}}{\rho - \frac{1}{\beta q\lambda}} \Rightarrow \frac{P_u}{P_\pi} = \frac{\frac{1}{\beta q}}{\rho - \frac{1}{\beta q\lambda}} = \frac{-\lambda}{1-\lambda\beta q\rho}$$

QED

Proposition A2: Optimal policy rule parameters formulas:

$$F_\pi = \varepsilon\,(P_\pi - 1) = \lambda\varepsilon P_\pi = \varepsilon\frac{\lambda}{1-\lambda} = \frac{1-\beta\lambda}{\kappa} \tag{18}$$
$$F_u = \varepsilon P_u = \varepsilon P_\pi \frac{\lambda}{\beta\lambda\rho - 1} = \varepsilon\frac{1}{1-\lambda}\frac{\lambda}{\beta\lambda\rho - 1} \tag{19}$$
$$\frac{F_u}{F_\pi} = A = \frac{1}{\lambda}\frac{P_u}{P_\pi} = \frac{1}{\beta\lambda\rho - 1} = \frac{P_u}{P_\pi - 1} = -1 + \beta\rho\frac{P_u}{P_\pi} \tag{20}$$

Demonstration:

The first order condition relates Lagrange multiplier to the policy instrument:

$$x_t = \varepsilon\gamma_{t+1} = \varepsilon(\gamma_t - \pi_t)$$
$$x_t = F_\pi\pi_t + F_u u_t = \varepsilon(\gamma_t - \pi_t) = \varepsilon(P_\pi\pi_t + P_u u_t - \pi_t) \Rightarrow$$
$$F_\pi = \varepsilon(P_\pi - 1),\; F_u = \varepsilon P_u$$

Proposition A3: From LQR to Gali (2015) vector basis (replace policy target by policy instrument).

One has:

$$\frac{-1-\kappa F_u}{\beta} = \frac{-1-\kappa A\frac{1-\beta\lambda}{\kappa}}{\beta} = \frac{1}{\beta\lambda\rho - 1}\lambda - \frac{1}{\beta} - \frac{\frac{1}{\beta\lambda\rho-1}}{\beta} = \frac{(1-\rho)\,\lambda}{\beta\lambda\rho - 1} = (1-\rho)\,\lambda A$$

One has:

$$\left\{\begin{array}{c}\begin{pmatrix} u_{t+1} \\ \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ (1-\rho) A\lambda & \lambda \end{pmatrix} \begin{pmatrix} u_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} \\ \begin{pmatrix} u_t \\ x_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ AF_\pi & F_\pi \end{pmatrix} \begin{pmatrix} u_t \\ \pi_t \end{pmatrix} = \mathbf{N} \begin{pmatrix} u_t \\ \pi_t \end{pmatrix} \\ x_t = F_\pi \pi_t + AF_\pi u_t \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{c}\begin{pmatrix} u_{t+1} \\ x_{t+1} \end{pmatrix} = \mathbf{N}^{-1} (\mathbf{A} + \mathbf{BF}) \mathbf{N} \begin{pmatrix} u_t \\ x_t \end{pmatrix} + \mathbf{N}^{-1} \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} \\ \begin{pmatrix} u_t \\ \pi_t \end{pmatrix} = \mathbf{N}^{-1} \begin{pmatrix} u_t \\ x_t \end{pmatrix} \\ \pi_t = \frac{1}{F_\pi} x_t - A\pi_t \end{array}\right.$$

One has:

$$\mathbf{N}^{-1} (\mathbf{A} + \mathbf{BF}) \mathbf{N} = \begin{pmatrix} \rho & 0 \\ (1-\lambda) F_\pi A\rho & \lambda \end{pmatrix}$$

Which is Gali (2015) representation of the solution:

$$x_t = \lambda x_{t-1} + (1-\lambda) F_\pi A\rho u_{t-1} = \lambda x_{t-1} + \varepsilon \frac{\lambda}{\beta q \lambda \rho - 1} \rho u_{t-1}$$

Proposition A4: Inequality demonstration.

One has the following inequalities

$$\text{If } 1 - \frac{1}{2}(\beta q + 1 + \kappa) + -\sqrt{\frac{1}{4}(\beta q)^2 \left(1 + \frac{1}{\beta q} + \frac{\kappa}{\beta q}\right)^2 - \beta q} > 0 \ \Leftrightarrow \sqrt{\frac{1}{4}(1 + \beta q + \kappa)^2 - \beta q} > \frac{1}{2}(-1 + \beta$$

$$\frac{1}{4}(1 + \beta q + \kappa)^2 - \beta q > \frac{1}{4}(-1 + \beta q + \kappa)^2$$

$$(1 + \beta q + \kappa)^2 - (-1 + \beta q + \kappa)^2 = 4(\kappa + \beta q) > 4\beta q$$

$$\kappa + \beta q > \beta q \text{ which is true.}$$