Increasing Returns, Balanced-Budget Rules, and Aggregate Fluctuations

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Abstract

This paper examines the conditions needed for equilibrium (in)determinacy in growth models characterized by perfect competition and a balanced-budget rule (BBR). Contrary to the literature that assumes zero public debt, we build an endogenous growth model with a generalized BBR authorizing a constant positive debt level. We show that the emergence of aggregate instability dramatically depends on the dynamics of the debt-to-GDP ratio and the strength of social labor externalities. If the labor demand is positively sloped and steeper than the labor supply, two reachable balanced-growth paths appear – a no growth trap, and a positive growth solution – that gives birth to both local and global indeterminacy, hence aggregate instability driven by self-fulfilling beliefs (sunspots). In addition, we show that a fiscal rule such that the tax rate strongly responds to public-debt increases can remove the no-growth trap, and secure positive long-run growth.

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1. Introduction

The emergence of indeterminacy in growth models characterized by perfect competition and a balanced-budget rule have received much attention in macroeconomic theory. In a seminal paper, Schmitt-Grohé and Uribe (1997, henceforth SU) first show that the steady state of a neoclassical growth model can be locally undetermined, giving rise to belief-driven fluctuations (sunspots). In their setup, public spending is exogenous, and the adjustment variable in the government's budget constraint is the tax rate on labor income.

Many recent contributions pursued the SU's research program by assuming a public spending-based adjustment with a fixed tax rate. Indeterminacy then crucially depends

on the usefulness of government expenditures. If public spending exerts some positive externality in households' utility function or in the production function, the steady state can remain undetermined.[2] However, if government expenditures are "wasteful", Guo and Harrison (2004, henceforth GH) showed that the steady state exhibits saddle-path stability, thus removing indeterminacy. Indeed, in an exogenous growth setup with a fixed tax rate on labor, the decreasing marginal products of capital and labor prevent agents expectations from becoming self-fulfilling.

The present paper explores the conditions for indeterminacy to appear in an endogenous growth framework, with other assumptions similar to SU and GH (in particular: a continuous-time one-sector model with endogenous labor, an additive utility function, and a balanced-budget rule, hereafter BBR). In this setup, we introduce two important ingredients. First, based on the seminal contribution of Benhabib and Farmer (1994), the aggregate production function exhibits increasing return-to-scale. With endogenous growth, the total factor productivity depends on the economy-wide levels of capital and labor through external effects in the production process (in the next section we detail several arguments that explain why labor social returns can be higher than the individual-firm returns). This can generate high labor externalities that produce nonlinearities in the dynamics of the consumption-to-capital ratio. Second, thanks to endogenous growth, our setup is characterized by non-trivial dynamics of the debt-to-capital ratio, even under the BBR. Effectively, the BBR defines situations such that public debt is constant, but non necessarily zero.[3] When the rate of economic growth is endogenously determined, such situations are consistent with either (i) a permanent positive growth giving rise to a (asymptotical) zero debt ratio in the long run, or (ii) a positive long-run debt ratio associated with the (asymptotical) disappearance of growth. This creates the possibility of multiple perfect-foresight balanced-growth paths (BGP).

Our results are as follows.
First, we show that, in an endogenous growth framework, wasteful government spending can be associated to both local and global indeterminacy. This finding conflicts with the GH's determinacy result established in a neoclassical growth model.
Second, we exhibit two regimes depending on the strength of social labor externalities.

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[2]See, e.g., Futagami and Mino (1995); Cazzavillan (1996); Palivos et al. (2003); Chen (2006); Guo and Harrison (2008); Chen and Guo (2013a,b), among others. Capital-labor substitution or consumption externalities can also generate indeterminacy in exogenous growth model, see, e.g. Alonso-Carrera et al. (2008) and Wong and Yip (2010).

[3]A number of recent works have shown that endogenous growth setups provide a useful framework for analyzing the effects of a continuous grow of public debt in the long run; see, e.g., Minea and Villieu (2012); Nishimura et al. (2015a); Boucekkine et al. (2015); Nishimura et al. (2015b); Menuet et al. (2018). Albeit we focus here on BBR regimes, to make our results easily comparable to GH, our model can be extended to deficit rules without qualitative changes (see our extension in subsection 4.2).

If externalities are low, only one steady state, characterized by positive growth, can be reached in the long run, and the model is locally and globally well-determined. In contrast, in the presence of high externalities, two reachable long-run equilibria emerge: a no-growth trap with positive debt ratio, and a positive-growth BGP with zero debt, the latter being undetermined. Our model is then characterized by aggregate instability, namely local and global indeterminacy, driven by self-fulfilling beliefs (sunspot). Importantly, the condition for indeterminacy to appear is the same as in Benhabib and Farmer (1994), namely the labor demand must be positively sloped and steeper than the labor supply. However, our aggregate-instability result covers a broader class of mechanisms than in Benhabib and Farmer (1994), because it may rely on local *and global* indeterminacy.

Third, we extend our model, and consider a fiscal rule such that the tax rate strongly responds to public-debt increases. Even if global indeterminacy remains (because two positive BGPs are feasible), this tax rule can remove the no-growth trap, and secure positive long-run growth. Indeed, a sufficient tax-increase can generate a positive relationship between government spending and public debt, such that the debt-to-output ratio follows an unstable trajectory around the no-growth trap.

Our paper complements the fast-growing literature on indeterminacy in endogenous growth models.[4] Starting from the seminal paper of Matsuyama (1991), local and global indeterminacy are mainly studied in two (or multiple)-sectors models,[5] or in the context of a public capital externality (productive or welfare-enhancing public spending).

In contrast, in our simple one-sector model with wasteful public spending, indeterminacy comes from two mechanisms. First, the presence of public debt is needed for multiplicity to arise.[6] Indeed, under a BBR, the non-trivial dynamics of the debt-to-capital ratio give rise to complex interactions between the government's budget constraint and the households' saving behavior. Second, following Benhabib and Farmer (1994); Farmer and Guo (1994), strong increasing returns-to-scale are needed for indeterminacy to emerge. In this case, labor supply and output positively depend on consumption, which can generate indeterminacy.

On the methodological side, the switch from exogenous to endogenous growth settings has two implications. One the one hand, the SU's and GH's conclusions regarding the determinacy of the steady state when the adjustment is based on wasteful public spending are dramatically changed.[7] On the other hand, in an endogenous growth framework, the

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[4]See the surveys of Benhabib and Farmer (1999), chap. 6, or Mino et al. (2008) regarding the local indeterminacy.

[5]See, e.g., Benhabib et al. (2000); Drugeon and Venditti (2001); Brito and Venditti (2010); Nishimura et al. (2013), among others.

[6]Some papers have shown that endogenous growth models with public debt generate two steady-states (Minea and Villieu, 2012; Nishimura et al., 2015a; Menuet et al., 2018).

[7]This issue has already been raised in the literature on progressive taxation, where neoclassical growth models are well determined (Guo and Lansing, 1998), while endogenous growth models (Chen and Guo,

Benhabib and Farmer (1994)'s indeterminacy condition is more realistic. Effectively, since the total factor productivity is endogenous, strongly increasing returns-to-scale (relative to all inputs) are reachable even if the production function exhibits constant returns-to-scale relative to reproducible factors.

In terms of policy implications, our results suggest that a public spending-based adjustment, together with a balanced-budget rule, can lead to an indeterminacy driven by agents' animal spirits. Then, the economy can converge to a no-growth trap in the long run. However, we show that this undesirable feature can be removed if the government adopts a fiscal rule such that the tax rate strongly responds to the increase in public debt. This result illustrates the so-called concept of "fiscal space", whereby public debt sustainability is ensured only if the primary budget surplus positively reacts to the debt burden (Ostry et al., 2015).

The remainder of the paper is organized as follows. Section 2 presents the model, section 3 studies a simple case without public debt, section 4 computes the steady states in the general case with public debt, section 5 deals with local and global dynamics, section 6 extends the model to a time-varying tax rate, and section 7 concludes.

2. The model

We consider a simple continuous-time endogenous-growth model with N representative individuals, and a government. Each representative agent consists of a household and a competitive firm. All agents are infinitely-lived and have perfect foresights. Population remains fixed over time, and we denote individual quantities by lower case letters, and aggregate quantities by corresponding upper case letters, namely $X = Nx$, for all variable X.

2.1. Households

The representative household starts at the initial period with a positive stock of capital (k_0), and chooses the path of consumption $\{c_t\}_{t\geq 0}$, hours worked $\{l_t\}_{t\geq 0}$, and capital $\{k_t\}_{t>0}$ to maximize the present discounted value of its lifetime utility, which is assumed to be separable

$$U = \int_0^\infty e^{-\rho t} \left\{ \log(c_t) - \frac{\chi}{1+\varepsilon} l_t^{1+\varepsilon} \right\} dt, \tag{1}$$

where $\rho \in (0, 1)$ is the subjective discount rate, $\varepsilon \geq 0$ is the constant elasticity of intertemporal substitution in labour, and $\chi > 0$ is a scale parameter.[8]

2013b) are associated with local indeterminacy.

[8]The specification corresponds to SU and GH.

Households use their income ($w_t l_t$, where w_t is the hourly wage rate), to consume (c_t), and to invest in capital ($\dot{k}_t$) and in government bonds (d_t), which return respectively q_t and r_t (the real interest rate). They pay taxes on the wage income ($\tau w_t l_t$, where τ is the exogenous wage tax rate), hence the following budget constraint

$$\dot{k}_t + \dot{d}_t = r_t d_t + q_t k_t + (1 - \tau) w_t l_t - c_t. \tag{2}$$

Defining by λ_t the costate variable of the current Hamiltonian, the first-order conditions for the maximization of the household's programme (with $r_t = q_t$ in competitive equilibrium) are

$$1/c_t = \lambda_t, \tag{3}$$

$$\chi l_t^\varepsilon = \lambda_t (1 - \tau) w_t,$$

and

$$\dot{\lambda}_t = (\rho - r_t) \lambda_t.$$

These conditions give rise to the familiar Keynes-Ramsey rule

$$\frac{\dot{c}_t}{c_t} = r_t - \rho, \tag{4}$$

and to the static relation

$$\chi l_t^\varepsilon = (1 - \tau) w_t / c_t. \tag{5}$$

Eq. (5) means that, at each period t, the marginal gain of hours worked (the net real wage $(1 - \tau) w_t$, expressed in terms of marginal utility of consumption $1/c_t$) just equals the marginal cost (χl_t^ε).

Finally, the optimal path of consumption has to verify the set of transversality conditions

$$\lim_{t \to +\infty} \{\exp(-\rho t) \, u'(c_t) \, k_t\} = 0 \text{ and } \lim_{t \to +\infty} \{\exp(-\rho t) \, u'(c_t) \, d_t\} = 0,$$

ensuring that lifetime utility U is bounded.

2.2. Firms

The representative firm produces output (y_t) using private capital (k_t) and labor (l_t), augmented by an externality that depends on the aggregate levels of both capital and labor, namely $y_t = \tilde{A} k_t^\alpha (l_t)^{1-\alpha} K_t^{\theta_1} L_t^{\theta_2}$, where $\tilde{A} > 0$ is a scale parameter, $\alpha \in (0, 1)$ is the elasticity of output to private capital, and θ_1 and θ_2 are positive parameters that reflect the externalities. This specification closely follows Benhabib and Farmer (1994).

As usual, the production function exhibits constant returns-to-scale at the individual level. Thus, the first-order conditions for profit maximization (relative to private factors)

are

$$r_t = \alpha \frac{y_t}{k_t}, \tag{6}$$

$$w_t = (1-\alpha)\frac{y_t}{l_t}. \tag{7}$$

To obtain an endogenous growth path, we consider constant capital returns ($\theta_1 = 1 - \alpha$). Thus, in the symmetric equilibrium ($L_t = Nl_t$, and $K_t = Nk_t$) we obtain

$$Y_t = \tilde{A}K_t L_t^\phi, \tag{8}$$

where $\phi := 1 - \alpha + \theta_2$.

In our model, labor social returns (ϕ) can be higher than the individual-firm returns $(1 - \alpha)$, thanks to the externalities. Three main arguments can sustain such a framework, and explain the presence of the parameter $\theta_2 \geq 0$: (i) knowledge externalities, (ii) agglomeration effects, and (iii) thick market externalities.

(i) First, following Romer (1986), one can consider that the output of an individual firm (y_t) is obtained with physical and human capital (h_t), the latter being produced with raw labor (or training activity) l_t and the economy-wide stock of knowledge X_t, namely $h_t = X_t l_t$. Using a constant returns-to-scale technology, we thus have $y_t = \tilde{A}k_t^\alpha h_t^{1-\alpha}$. Suppose that knowledge is produced by a simple Cobb-Douglas technology depending on the aggregate levels of physical and human capital: $X_t = H_t^\beta K_t^{1-\beta}$, where $\beta \in (0,1)$ is a measure of human capital efficiency in the accumulation of knowledge. At the aggregate level, we then obtain $H_t = K_t L_t^{1/(1-\beta)}$.[9] Hence, the aggregate production function is $Y_t = \tilde{A}K_t L_t^\phi$, with $\phi = (1-\alpha)/(1-\beta)$. Then, parameter θ_2 is endogenously determined as $\theta_2 = \beta(1-\alpha)/(1-\beta) \geq 0$.

(ii) Second, agglomeration effects can justify Eq. (8). For example, urban production externalities are likely to produce spillovers among producers in areas with a high density of economic activity (Krugman, 1991). In the presence of high localized knowledge spillovers, firms tend to locate in the proximity of each other to capitalize on the aggregate knowledge stock, namely, e.g., $y_{tj} = \tilde{A}k_t^\alpha (l_t e_{jt})^{1-\alpha}$, where e_{jt} is the agglomeration externality in location j in time t. Following Lucas and Rossi-Hansberg (2002), e_{jt} measures the mass of firm in location j, namely, if the size of the agglomeration is assumed to be one, $e_{tj} = \int_0^1 y_{tj}dj$. In symmetric equilibrium, $y_{tj} = y_t$ for any j, the aggregate production function becomes $Y_t = \tilde{A}K_t L_t^{(1-\alpha)/\alpha}$. In this case, $\phi = (1-\alpha)/\alpha$, and the

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[9] Human capital externalities, i.e. the mere idea that your coworkers' human capital makes you more productive, are well documented in the empirical literature (see, e.g. Rauch, 1993; Moretti, 2004, who provide robust support for large human capital externalities). Note that, since knowledge is produced with both human and physical capital, labor externalities can be very high in the aggregate production function (8). Alternative models of endogenous growth, based on the Lucas (1988)'s archetype, consider the formation of human capital through individual training decisions that compete with productive activities. See also Bond et al. (1996).

corresponding value obtained for θ_2 is now $\theta_2 = (1-\alpha)^2/\alpha \geq 0$.

(iii) Third, another source of labor externalities may be derived from "thick market externalities" (Diamond, 1982), namely a friction related to firm-to-firm transactions. This externality is based on the increasing returns-to-scale matching technology in labor markets, so that search costs are lower as the size of the labor market increases.[10] The adoption of an innovation can also generate a positive externality on other traders, thus creating strategic complementarities between the innovation decisions of different firms (Acemoglu, 1997). Altogether, these arguments support the presence of labor returns, which are potentially higher from the social perspective than from the firm-specific level (i.e. $\phi > 1 - \alpha$).

2.3. The government

The government provides public expenditures G_t, levies taxes, and borrows from households. The fiscal deficit is financed by issuing debt ($\dot{D}_t$), hence, the following budget constraint

$$\dot{D}_t = r_t D_t + G_t - \tau w_t L_t. \tag{9}$$

With a fixed tax rate (τ), there are two potential policy instruments in Eq. (9): public debt (D_t), and public spending (G_t). Following GH, we consider endogenous public spending under a balanced-budget rule (BBR). Without public debt, this rule is simply $G_t = \tau w_t L_t$. However, with public debt, the BBR requires $\dot{D}_t = 0 \Rightarrow D_t = D_0, \forall t$, namely

$$G_t = \tau w_t L_t - r_t D_0. \tag{10}$$

Hence; the BBR is consistent with a constant (not necessarily zero) public debt level. We denote this relation as a *generalized* BBR. Such a situation characterizes almost all countries under BBR, because, at the date they adopt the BBR, countries most often experiment high public debt ($D_0 > 0$). This feature has crucial implications, since our model can exhibit complex dynamics of the public debt *ratio* (D_0/Y_t), even in the presence of a BBR.[11]

2.4. Equilibrium

To find endogenous growth solutions, we deflate all growing variables by the capital stock to obtain long-run stationary ratios, namely (we henceforth omit time indexes) $y_k := Y/K$, $c_k := C/K$ and $d_k := D_0/K$.

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[10]Following Hall (1993) "*the overall technology of an industry with a thick-market externality will have increasing returns even though each firm has constant returns*" (p. 94). Caballero and Lyons (1992) found empirical support for external increasing returns to scale, which can be interpreted as thick-market externalities. For a growth model with endogenous human capital and labor participation, see Chen et al. (2011).

[11]Our results hold with a more general set of deficit rules, such as, e.g., $\dot{D}_t = \theta Y_t$, where θ is the constant deficit-to-GDP ratio. See the subsection 4.2 for a detailed analysis.

The optimal consumption behaviour is, from (4) and (6),

$$\frac{\dot{C}}{C} = \alpha y_k - \rho, \tag{11}$$

while the path of the capital stock is given by the goods market equilibrium

$$\frac{\dot{K}}{K} = y_k - c_k - g_k. \tag{12}$$

From (6) and (10), the public spending ratio is defined as

$$g_k = [\tau(1 - \alpha) - \alpha d_k]y_k, \tag{13}$$

and from (5) and (7) we obtain the output ratio

$$y_k = Ac_k^{\psi}, \tag{14}$$

where $\psi := \phi/(\phi - 1 - \varepsilon)$, and $A := \tilde{A} \left[\frac{\chi}{\tilde{A}(1-\alpha)(1-\tau)N^{\varepsilon}} \right]^{\psi} > 0$.

The relationship between the consumption and the output ratios crucially depends on the sign of ψ, which is related to the strength of labor social returns. Effectively, the aggregate production function can exhibit increasing labor return if labor market externalities are high enough (i.e. $\theta_2 > \alpha \Rightarrow \phi > 1$). Consequently, ψ can be either positive – and higher than one –[12] or negative, thereby depicting an increasing or a decreasing relation between the consumption and the output ratios.

The reduced-form of the model is obtained from Eqs. (6)-(7)-(10)–(11)-(12)-(13), namely, using $\bar{d} := (1 - \tau)(1 - \alpha) \in (0, 1)$,

$$\begin{cases} \dfrac{\dot{c}_k}{c_k} = c_k - \rho - (\bar{d} + \alpha d_k)y_k, \\[4mm] \dot{d}_k = [c_k - (\bar{d} + \alpha + \alpha d_k)y_k]d_k. \end{cases} \tag{15}$$

As it should clearly appear, the presence of a constant positive public debt level in the generalized BBR (10) generates non trivial dynamics of the public-debt *ratio* (d_k) in the reduced form (15). In an exogenous growth context, the generalized BBR (10) does not fundamentally change the dynamic properties of the steady state compared with a simple BBR without debt, because the steady-state is unique and well determined, just as in GH (see Appendix A). With endogenous growth, in contrast, the generalized BBR, if associated to strong increasing returns in the production function, can produce multiple

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[12]Observe that $\psi > 0 \Rightarrow \psi > 1$. Indeed, as $\phi > 0$, the image of function $\phi \mapsto \phi/(\phi - 1 - \varepsilon)$ is $\mathbb{R} \setminus (0, 1)$.

perfect-foresight BGPs with possible local and global indeterminacy, as we will see.

We determine the steady-state solutions of the model in section 4, and analyze local and global dynamics in section 5. Beforehand, to isolate the specific role of increasing returns and provide some intuition on the underlying mechanisms of the model, section 3 presents a simple special case without public debt.

3. A preliminary analysis: the case without public debt

Without public debt ($d_k = \dot{d}_k = 0$), the BBR simply writes $G_t = \tau w_t L_t$, and, using (13), the reduced-form of the model (15) boils down to a unique equation

$$\frac{\dot{c}_k}{c_k} = c_k - \rho - \bar{d} A c_k^{\psi} =: \Psi(c_k), \tag{16}$$

such that the long-run consumption ratio is defined by the implicit relation $\Psi(c_k) = 0$.

If $\psi < 0$, $\Psi(\cdot)$ is continuously increasing, hence the existence of one unique positive solution (point P in Figure 1a). Moreover, as $\Psi'(\cdot) > 0$, this solution is unstable. Since the consumption ratio c_k is a jump variable, the steady state is well-determined (the economy initially jumps at point P).

However, if $\psi > 1$, $\Psi(\cdot)$ takes a maximum at point $\hat{c}_k := (\psi \bar{d} A)^{1/(1-\psi)}$, such that $\Psi'(\hat{c}_k) = 0$ and $\Psi''(\hat{c}_k) < 0$. Thus, there are two positive long-run solutions, namely c_k^N and c_k^P, which correspond respectively to points N and P in Figure 1b. Note that c_k^N is associated to a negative growth rate in steady state, while c_k^P is associated to a positive BGP, under the mild condition that the discount rate is small enough, i.e. $\rho < \bar{\rho} := \alpha A [A(\alpha + \bar{d})]^{\psi/(1-\psi)}$, that we assume throughout the paper (as we will see below, this condition ensures the existence of BGPs).[13] Clearly, as $\Psi'(c_k^N) > 0$ and $\Psi'(c_k^P) < 0$, and c_k being a jump variable, the negative growth solution is unstable (well determined) while the positive growth solution is saddle-point stable (locally undetermined).
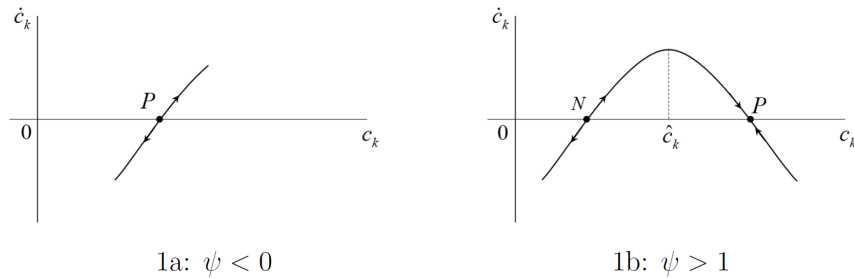

1a: $\psi < 0$ 1b: $\psi > 1$

Figure 1: Phase portrait in the no-debt case

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[13]In the long run, the growth rate is $\gamma = \alpha A c_k^{\psi} - \rho$. Thus, since $\psi > 1$, $\gamma > 0 \Leftrightarrow c_k > \bar{c}_k := (\rho/\alpha A)^{1/\psi}$. Yet, $\Psi(\bar{c}_k) > 0 \Leftrightarrow \rho < \bar{\rho}$. Consequently, if $\rho < \bar{\rho}$, there are two positive solutions: $c_k^N < \bar{c}_k$ and $c_k^P > \bar{c}_k$; hence, the long-run growth rate is negative at point N and positive at point P.

Consequently, in the case $\psi > 1$ the economy is characterized by aggregate instability that relies both on local and global indeterminacy. Effectively, two perfect-foresight BGPs can be reached in the long-run (global indeterminacy), and the transition path in the neighborhood of P is subject to sunspot equilibria (local indeterminacy). If $\psi < 0$, in contrast, the BGP is unique and there is no aggregate instability, as in GH.

The existence of these two regimes crucially rests on the relation between the consumption and the output ratios in Eq. (14), which can be either positive or negative. This property comes from the labor market equilibrium (5). As consumption increases, its marginal utility decreases, thus inducing the representative household to substitute leisure for working hours, because the shadow price of labor (λ, see Eq. 3) declines. The higher the elasticity (ε) the stronger the reduction in worked hours. To restore equilibrium, the real wage must rise. In the first order condition for profit maximization (7), this arises only if the average product of labor (Y/L) goes up, namely if $\phi > 1$. Therefore, as the consumption ratio jumps up, households are induced to increase labor supply only if the real wage responds strongly enough to compensate the disutility of work ($\phi > 1 + \varepsilon \Rightarrow \psi > 1$); hence a rise in the output ratio. In the opposite case ($\phi < 1 + \varepsilon \Rightarrow \psi < 0$), labor supply decreases and the output ratio goes down.

The intuition of this analysis is close to Benhabib and Farmer (1994), but in the context of endogenous growth. By injecting the real wage rate (7) into the first-order condition for the optimal choice of labor (5), we obtain

$$\chi(C_t/K_t)(L_t/N)^\varepsilon = A(1-\tau)(1-\alpha)L_t^{\phi-1}.$$

By taking logarithms, the labor supply curve is

$$\tilde{w}_t = \text{cst}_1 + \ln(C_t/K_t) + \varepsilon\tilde{L}_t,$$

and the labor demand curve is

$$\tilde{w}_t = \text{cst}_2 + (\phi - 1)\tilde{L}_t.$$

where $\tilde{w}_t$ is the logarithm of the real wage per capital, and $\tilde{L}_t$ the logarithm of the real labor (with the two constants $\text{cst}_1 := \ln(\chi/N^\varepsilon)$ and $\text{cst}_2 := \ln(A(1-\tau)(1-\alpha))$). Thus, labor demand is decreasing with real wage if and only if $\phi < 1$. In this case, the higher the consumption ratio, the higher the employment and the output on the BGP. In contrast, if $\phi > 1$, labor demand is, in equilibrium, positively-sloped.

Figure 2 examines the effects of a rise in the consumption ratio in the case of a positively-sloped labor demand ($\phi > 1$). If $\varepsilon < \phi - 1$ (i.e. $\psi > 0$), the slope of the demand curve ($\phi - 1$) exceeds the slope of the supply curve (ε), so that an increase in the consumption-to-capital ratio rises employment, output, and the real wage (see Figure 2a). In contrast, if $\varepsilon > \phi - 1$ (i.e. $\psi < 0$), a rise in the consumption ratio is accompanied

by a decline in employment and output on the BGP (Figure 2b illustrates this case for $1 < \phi < 1 + \varepsilon$).
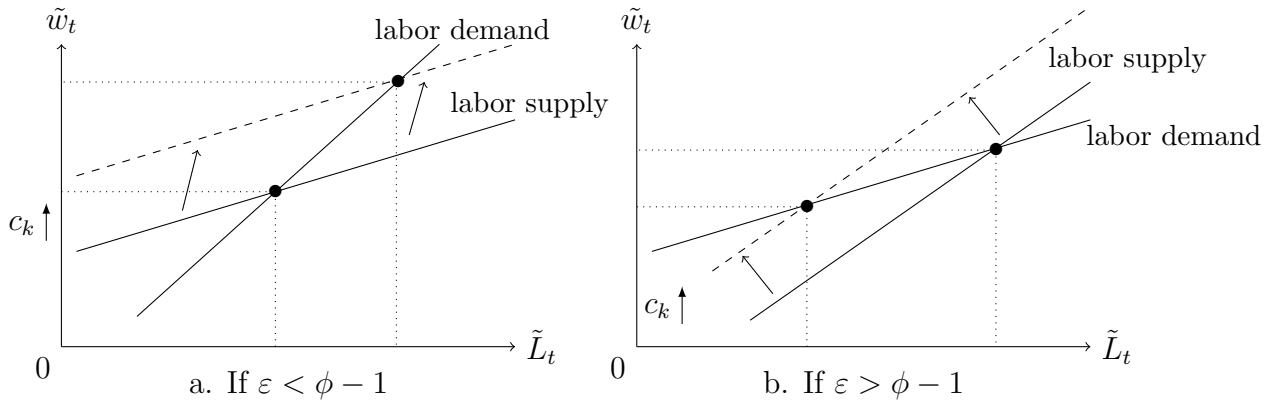

Figure 2: Effect of a rise in the consumption ratio (positively sloped labor demand)

Importantly, the condition $\psi > 0$ corresponds to the Benhabib and Farmer (1994)'s criterion to generate local indeterminacy in a neoclassical growth model with increasing returns. In our model too, a necessary condition for aggregate instability to emerge is that the real wage is increasing with the employment level.

Two points deserve particular attention. First, although this criterion has been challenged from an empirical perspective,[14] in our endogenous growth setup the condition $\phi > 1 + \varepsilon$ is realistic. Effectively, considering the knowledge-externality interpretation of the model (with the firm-specific and knowledge production functions $y_t = \tilde{A}k_t^\alpha(X_t l_t)^{1-\alpha}$ and $X_t = H_t^\beta K_t^{1-\beta}$, respectively), this condition is verified for, e.g., $\beta > \alpha$ if the Frisch elasticity of labor supply is infinite ($\varepsilon = 0$), as usual in business cycle literature (see, e.g. SGU). It looks as quite reasonable to suppose that the elasticity of knowledge to human capital is greater than the physical capital share in output (around $\alpha = 1/3$).

Second, in our model, the criterion $\phi > 1 + \varepsilon$ is a necessary and sufficient condition to ensure not only local but also *global* indeterminacy. Nevertheless, at this stage our

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[14]To this day, there is no agreement in the literature on the degree of aggregate returns-to-scale. The first papers based on simple linear regressions in macro models highlighted quantitatively significant increasing returns-to-scale in U.S. manufacturing (Caballero and Lyons, 1992; Hall, 1993). Subsequent works challenged this view (Basu and Fernald, 1997; Burnside, 1996), suggesting that the U.S. manufacturing industry displays constant returns with no external effects (while there is significant heterogeneity across industries). However, more recent papers found that, contrary to many previous results, US economic growth has been driven by increasing returns-to-scale rather than technical progress (Diewert and Fox, 2008). As pointed out by Hall (1993), this controversy can be related to the difficulty of identifying separately the returns to scale and TFP growth. In neoclassical (exogenous) growth models, TPF is the engine of growth, and the measure of returns-to-scale must be insolated from the measure of technical progress. In our endogenous growth setting, TFP is endogenous, and the production function exhibits constant returns-to-scale relative to reproducible factors, i.e. when labor is measured in efficient units ($y_t = \tilde{A}k_t^\alpha h_t^{1-\alpha}$). This does not preclude the existence of large increasing returns-to-scale at the aggregate level, associated to endogenous technical change.

setup exhibits two shortcomings. Effectively, the absence of transitory dynamics and the presence of a negative long-run BGP are questionable, both on the empirical side and from an endogenous growth perspective. The following sections address these issues, by considering the general version of the model with public debt.

4. The general case with public debt

In the general model, we define a BGP as a path on which consumption, capital, and output grow at the same (endogenous) rate, namely: $\dot{c}_k = \dot{d}_k = 0$ in (15). Thus, for any steady-state i, we have: $\gamma^i := \dot{C}/C = \dot{K}/K = \dot{Y}/Y$, while the real interest rate (r^i) is constant.

The long-run endogenous growth solutions can be described by two relations between c_k and d_k. The first one is the $\dot{c}_k = 0$ locus, which comes from the Keynes-Ramsey relation (11) and the IS equilibrium (12)

$$c_k - \rho = (\bar{d} + \alpha d_k)y_k. \tag{17}$$

The second relation is the $\dot{d}_k = 0$ locus, namely

$$\gamma d_k = 0, \tag{18}$$

where $\gamma := (\bar{d} + \alpha + \alpha d_k)y_k - c_k$ is the long-run growth rate. Eq. (18) means that either the long-run public debt is positive and the associated growth rate is zero (*zero growth solution*, such as $d_k > 0 \Rightarrow \gamma = 0$), or the long-run growth rate is positive (or negative) with zero public debt ($\gamma > (<)0 \Rightarrow d_k = 0$).

4.1. Characterization of steady states

Steady-state solutions are obtained at the crossing-points of Eqs. (17) and (18).

Theorem 1. *If $\rho < \bar{\rho}$, the long-run equilibria are characterized by the following regimes.*

- *Regime $\mathcal{R}_1$: $\psi < 0$. There are one positive growth solution (point Q), and one zero growth solution (point Z).*

- *Regime $\mathcal{R}_2$: $\psi > 0$. There are one negative solution (N), one positive solution (P), and one zero growth solution (point Z).*

Proof. See Appendix B.

The multiplicity of steady states comes from the interaction between two nonlinear relationships linking the consumption and the debt ratios.

(i) The first relation stems from the government's budget constraint $\dot{d}_k = -\gamma d_k = 0$, which is verified either if $d_k = 0$ (namely the x-axis in Figures 3 and 4) or if

$$\gamma = (\bar{d} + \alpha + \alpha d_k)y_k(c_k) - c_k = c_k[(\bar{d} + \alpha + \alpha d_k)Ac_k^{\psi-1} - 1] = 0.$$

Considering strictly-positive consumption ratios, this relation describes a monotonic association between c_k and d_k, which is positively-sloped if $\psi < 0$ (Figure 3), and negatively-sloped if $\psi > 0 \Rightarrow \psi > 1$ (Figure 4).

(ii) The second relation is derived from the assumption of balanced-growth in the long run (namely $\dot{K}/K = \dot{C}/C = \gamma$), and corresponds to (17), with $y_k(c_k) = Ac_k^{\psi}$.

If $\psi < 0$, an increase in the consumption ratio (c_k) reduces the output ratio (y_k), decreasing the RHS of (17). This generates a monotonic increasing relation between c_k and d_k (any positive deficit ratio is associated with one consumption ratio), as depicted in Figure 3. Therefore, there is only one consumption ratio that verifies $d_k = 0$, hence the uniqueness of the non-zero growth solution (point Q).
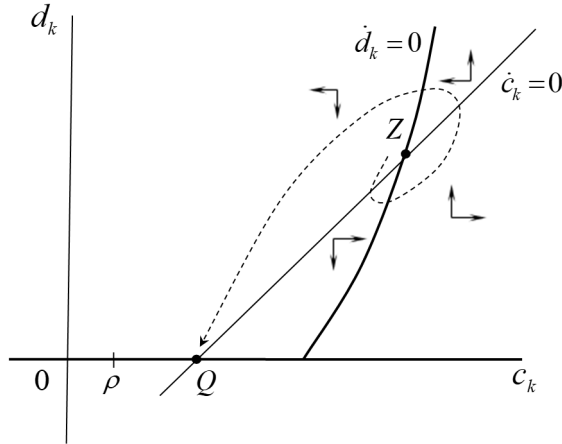

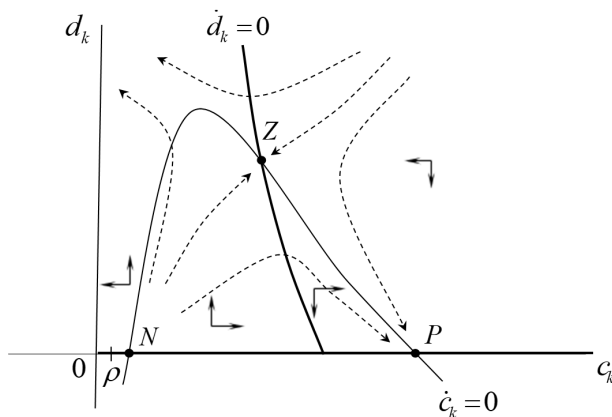Figure 3: Phase portrait under regime $\mathcal{R}_1$

Figure 4:   Phase portrait under regime $\mathcal{R}_2$

In contrast, if $\psi > 0$ (see Figure 4), output increases with the consumption ratio, so that the two sides of Eq. (17) positively depend on c_k, producing a non-monotonic relationship between c_k and d_k. In this case, any positive deficit ratio is associated with two consumption ratios. Consequently, two levels of c_k verify $d_k = 0$, hence the two non-zero growth solutions (points N and P).

4.2. Extension to permanent deficit rules

Although we focused so far on the BBR, our setup can be extended to the presence of continuous growth of public debt in the long run. Indeed, a number of recent papers have shown that endogenous growth is feasible with growing public debt in the steady state.[15] With endogenous growth, output grows continuously along the BGP, allowing public debt to grow continuously (since the BGP is such that all variables grow at the same rate, the debt-to-output ratio is constant in the steady-state). Thus, contrary to exogenous growth models, the BBR is not necessarily required.

Hence, our model can be extended to deficits rules involving a constant deficit-to-output ratio $\theta \geq 0$, such that[16]

$$\dot{D}_t = \theta Y_t. \tag{19}$$

In this respect, the case $\theta = 0$ amounts to our generalized BBR, which corresponds to a constant (but not necessarily zero) public debt.

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[15]See, e.g., Minea and Villieu (2012), Nishimura et al. (2015a), Nishimura et al. (2015b), and Menuet et al. (2018) for an extension to money financing.

[16]This rule authorizes permanent deficits in the long run, which is consistent with OECD's fiscal positions (prior to the Great Recession, the average public deficit was about 2.5% of GDP from 1970 to 2005, namely $\theta = 0.025$). One could also specify a deficit rule with a gradual adjustment path of the deficit-to-output ratio to a long-run target without qualitative change in the model.

With such a deficit rule, the reduced form of the model (15) must be amended as follows: (i) coefficient $\bar{d}$ must be redefined as $\bar{d} := (1 - \tau)(1 - \alpha) - \theta$, which is positive for small θ, and (ii) the debt to capital ratio now evolves according to

$$\dot{d}_k = \theta y_k + [c_k - (\bar{d} + \alpha + \alpha d_k)y_k]d_k.$$

Consequently, the $\dot{c}_k = 0$ locus (17) is unchanged, notwithstanding the redefinition of $\bar{d}$, while the $\dot{d}_k = 0$ locus (18) becomes now

$$\theta y_k = \gamma d_k. \tag{20}$$

With small positive values of the deficit ratio (namely $\theta > 0$), previous results are qualitatively unchanged, as show Figures 5 and 6, where the dashed- (resp. continuous-) bold lines describe the $\dot{d}_k = 0$ locus in the case $\theta = 0$ (resp. $\theta > 0$). The major change is that the $\dot{d}_k = 0$ locus is now a smooth curve that intersects twice the $\dot{c}_k = 0$ locus, with point Z being now characterized by positive growth. Hence, the deficit rule $\theta > 0$ precludes negative and zero growth solutions in the steady state, because the long-run economic growth rate must be $\gamma = \dot{D}/D = \theta Y/D > 0$ in equilibrium. For the same reason, the no-growth trap Z becomes now a *low*-growth trap, with (small) positive economic growth in equilibrium.
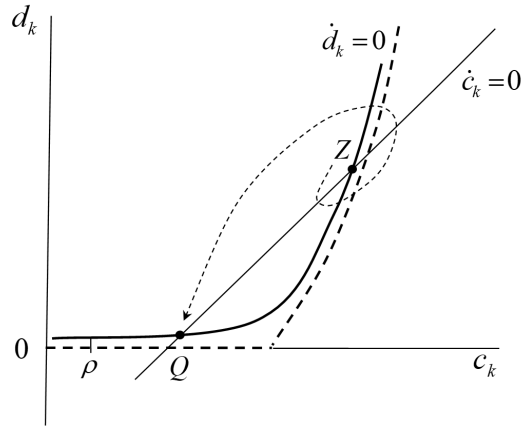

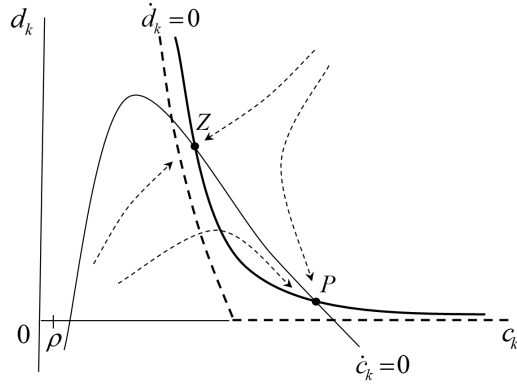Figure 5: Phase portrait under regime $\mathcal{R}_1$ $(\theta > 0)$

Figure 6: Phase portrait under regime $\mathcal{R}_2$ ($\theta > 0$)

5. Analysis of dynamics

To study the behaviour of the model outside the steady-state, we first proceed to a local stability analysis, and then infer global dynamics. To simplify the exposition, we focus on the (generalized) BBR case ($\theta = 0$).[17]

5.1. Local dynamics

By linearization in the neighborhood of steady-state i, $i \in \{N, P, Q, Z\}$, the system (15) behaves according to $(\dot{c}_k, \dot{d}_k) = \mathbf{J}^i(c_k - c_k^i, d_k - d_k^i)$, where $\mathbf{J}^i$ is the Jacobian matrix. The reduced-form includes one jump variable (the consumption ratio c_{k0}) and one pre-determined variable (the public-debt ratio d_{k0}, since the initial stocks of public debt D_0 and private capital K_0 are predetermined). Thus, for BGP i to be well determined, $\mathbf{J}^i$ must have two opposite-sign eigenvalues. Using (15), we compute

$$\mathbf{J}^i = \begin{pmatrix} CC^i & CD^i \\ DC^i & DD^i \end{pmatrix},$$

where

$$CC^i = c_k^i - (\bar{d} + \alpha d_k^i)\psi y_k^i, \tag{21}$$

$$CD^i = -\alpha c_k^i y_k^i, \tag{22}$$

$$DD^i = -\gamma^i - \alpha d_k^i y_k^i, \tag{23}$$

$$DC^i = [1 - A\psi(c_k^i)^{\psi-1}(\bar{d} + \alpha + \alpha d_k^i)]d_k^i. \tag{24}$$

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[17]The behaviour of the model is qualitatively unchanged with a deficit rule ($\theta > 0$), provided that the deficit ratio is small (formally $\theta \to 0$).

Hence, the trace and the determinant of the Jacobian matrix are, respectively

$$\text{Tr}(\mathbf{J}^i) = c_k^i - (\bar{d} + \alpha d_k^i)\psi y_k^i - \gamma^i - \alpha d_k^i y_k^i, \tag{25}$$

$$\det(\mathbf{J}^i) = -\gamma^i[c_k^i - (\bar{d} + \alpha d_k^i)\psi y_k^i] - \psi\alpha^2(y_k^i)^2 d_k^i. \tag{26}$$

The following theorem establishes the topological behaviour of each steady-state.

Theorem 2. *(Local Stability)*

- *In regime $\mathcal{R}_1$, Z is locally over-determined (unstable), and Q is locally determined (saddle-point stable).*

- *In regime $\mathcal{R}_2$, Z is locally determined (saddle-point), P is locally under-determined (stable), and N is locally over-determined (unstable).*

Proof. See Appendix C.

The local stability of the zero growth solution (Z) is intuitive. Indeed, the sign of the determinant is the opposite of the sign of ψ. Therefore, in regime $\mathcal{R}_2$ ($\psi > 0$), the determinant is negative, hence the saddle-path property. In regime $\mathcal{R}_1$ ($\psi < 0$), both the determinant and the trace are positive, hence the point Z is unstable.

As regards the non-zero growth solutions, we have $d_k^i = 0$, namely $DC^i = 0$, $i \in \{P, N, Q\}$. Consequently, the two eigenvalues (λ_1^i, λ_2^i) are simply $\lambda_1^i = DD^i = -\gamma^i$, and $\lambda_2^i = CC^i = c_k^i - (\bar{d} + \alpha d_k^i)y_k^i$. In regime $\mathcal{R}_1$, point Q is associated to a positive growth rate ($\gamma^Q > 0$), and to $CC^Q > 0$, hence the law of motion of consumption is unstable. Consequently, there is one negative and one positive eigenvalue, and Q is saddle-path stable. In regime $\mathcal{R}_2$, in contrast, the positive growth solution is such that $CC^P < 0$. The law of motion of consumption is then stable, the two eigenvalues are negative, and P is locally undetermined (stable). At the negative growth solution (point N), however, we have $CC^N > 0$, so that the two eigenvalues are positive, which makes N a locally over-determined (unstable) solution.

5.2. Global dynamics

Capitalizing on the local analysis, we can now evaluate the global dynamics. According to theorem 2, we can distinguish two cases, depending on the sign of ψ.

Regime $\mathcal{R}_1$ – In this case, there are two steady states, but only the positive growth solution (Q) can be reached in the long run, as the zero growth trap (Z) is unstable. Indeed, for any predetermined debt ratio $d_{k0} > 0$ the consumption ratio c_{k0} jumps to place the economy on the saddle-path that converges towards Q, which defines the unique long-run equilibrium (hence the absence of local or global indeterminacy). Consequently, in an endogenous growth setup with wasteful public spending, we find the GH's determinacy result. However, this result does not hold in regime $\mathcal{R}_2$.

Regime $\mathcal{R}_2$ – As we have seen, there are three steady states: N is unstable, P is stable, and Z is saddle-path stable. Thus, the situation is characterized by local indeterminacy (in the vicinity of the positive growth solution P), and global indeterminacy. Indeed, given a (predetermined) public debt ratio $d_{k0} > 0$, the initial consumption ratio can jump on the unique transition path that converges towards Z, or on one of the multiple paths that converges towards P.

Consequently, the long-run equilibrium of the economy is subject to "animal spirits", in the form of optimistic or pessimistic views of households at the initial time. The intuition is as follows. Let us suppose that, at the initial time, households expect a high consumption ratio and zero debt (point P) in steady state. Thus, the output ratio is expected to be high (as $\psi > 0$), which corresponds to high expected return of private capital. Then, at the initial time households are induced to save, leading to a self-fulfilling increase in economic growth. As a result, a belief-driven transitional path towards the the positive growth BGP (P) emerges.

In contrast, following the same mechanism, if high public debt and low return of saving are expected (along the Z BGP), households increase consumption and reduce investment at the initial time. This puts the economy on a self-fulfilling path towards the no-growth trap.

5.3. Discussion

In our model, multiplicity and indeterminacy result from two ingredients that closely interact: the dynamics of the public debt ratio, and the dynamics of the consumption ratio.

Regarding the first ingredient (ignoring negative growth situations), the BBR generates two possible BGPs: a zero-growth solution with positive debt, and a zero debt solution with positive growth. Regarding the second ingredient, the determinacy of these two equilibria fundamentally depends on the effect of the consumption ratio in the IS equilibrium. As we have seen, in goods market, consumption increases the demand, while its effect on supply is more complex and depends on the labor market equilibrium through the sign of ψ.

If $\phi < 1 + \varepsilon \Rightarrow \psi < 0$, an increase in consumption negatively affects output, because households are induced to increase leisure. Consequently, a jump in the consumption ratio (c_k) generates an excess demand in the goods market, producing an adverse effect on capital accumulation ($\dot{K}/K$). This, leads to an increase in the path of the consumption ratio ($\dot{c}_k/c_k$). Hence, the dynamics of c_k is unstable, which explains the absence of indeterminacy.

If $\phi > 1 + \varepsilon \Rightarrow \psi > 1$, following a positive jump in consumption, the rise in the real wage is strong enough to overcome the disutility of labor, thereby labor supply and output increase. Interestingly, since $\psi > 1$, this supply effect outweighs the increase in demand. As a result, the increase in the consumption ratio now generates a net *excess supply* in

the goods market. This boosts capital accumulation, which dampens the evolution of the consumption ratio ($\dot{c}_k/c_k$ is reduced). Hence, the dynamic of c_k now becomes stable, giving birth to indeterminacy.

Roughly speaking, our model is in some sense "more stable" in the presence of strong labor externalities (i.e. $\psi > 0$). This is consistent with the Benhabib and Farmer (1994)'s criterion to generate local indeterminacy of the perfect foresight equilibrium in a neoclassical growth framework. In our endogenous growth setup too, indeterminacy only emerges if $\psi > 0$, but this indeterminacy presents a larger perspective than in Benhabib and Farmer (1994). Effectively, the case $\psi > 0$ leads not only to local indeterminacy but also to a multiplicity of equilibria, leading to *global* indeterminacy. Indeed, for any initial value of the predetermined variable (the deficit ratio), the economy can reach either a positive growth solution, or an undesirable stagnation solution with zero growth, according to optimistic or pessimistic views of households. Given the undesirable possibility of the zero growth in the long-run, in the following section we relax the assumption of a constant tax rate, and attempt to characterize suitable fiscal rules that allow escaping the no-growth trap.

6. How to remove the no-growth trap? The case with a time-varying tax rate

So far, we have considered a fixed tax rate. In this section, we extend our analysis to the case of a time-varying tax rate. To this end, we assume that the government is subject to a tax rule $\tau(d_k)$, which belongs to the class of tax rules ($\mathcal{T}$) requiring a higher tax rate as the public debt increases, namely $\mathcal{T} := \{\tau(d_k)|\tau(d_k) \in \mathcal{C}^2, \tau'(d_k) \geq 0\}$. To assess whether such a fiscal rule allows removing the no-growth solution, we focus from now on regime $\mathcal{R}_2$.

It must be emphasised that, even if the tax rate is variable, the fiscal rule $\tau(d_k)$ remains exogenously specified, and the adjustment variable in the government's budget constraint remains wasteful public spending (g_k), as in the previous sections.

The dynamic system (15) becomes

$$\begin{cases} \dfrac{\dot{c}_k}{c_k} = c_k - \rho - (\bar{d}(d_k) + \alpha d_k)y_k, \\[2mm] \dot{d}_k = [c_k - (\bar{d}(d_k) + \alpha + \alpha d_k)y_k]d_k, \end{cases} \tag{27}$$

where $\bar{d}(d_k) = (1 - \tau(d_k))(1 - \alpha) \in (0, 1)$.

With a time-varying tax rate, the relationship between public debt and public spending is modified. Effectively, as $g_k = rd_k + \tau(d_k)w_t L_t = [\tau(d_k)(1 - \alpha) - \alpha d_k]y_k$, we can distinguish two cases following an increase in the debt ratio (d_k).

(i) If $\tau'(d_k) < \hat{\tau} := \alpha/(1 - \alpha)$, the increase in the debt burden exceeds the new fiscal revenues, forcing public spending to decline, i.e. $\partial g_k/\partial d_k < 0$. The situation is

qualitatively unchanged relative to the previous sections.

(ii) In contrast, if the fiscal rule generates high tax revenues, namely $\tau'(d_k) > \hat{\tau}$, fiscal resources are strong enough to offset the increase in the debt burden, so that public expenditures rise, i.e. $\partial g_k / \partial d_k > 0$. In this case, the dynamics of the model radically changes, as shows the following theorem.

Theorem 3. *In regime $\mathcal{R}_2$ (i.e. $\psi > 0$), if $\tau' > \hat{\tau}$, there are critical levels $\hat{\rho} > 0$ and $\bar{\tau}_0 > 0$, such that, if $\rho < \hat{\rho}$ and $\tau(0) < \bar{\tau}_0$, there are two positive growth solutions (M and H), and one zero growth solution (Z).*

Proof. See Appendix D.

The three equilibria still come from the interaction between two nonlinear relationships linking the consumption and the deficit ratios. If $\tau' < \hat{\tau}$, the situation is unchanged compared with the theorem 1. However, if $\tau' > \hat{\tau}$, two positive growth solutions emerge. Effectively, regarding the phase portrait (see Figure 7),[18] there are two main differences relative to the previous section.

First, the stationary locus $\dot{c}_k = 0$ now describes a U-shaped curve in the (c_k, d_k)-plane, and corresponds to

$$c_k - \rho = \zeta(d_k) y_k(c_k), \quad \text{where } \zeta(d_k) := \bar{d}(d_k) + \alpha d_k. \tag{28}$$

As $\tau'(d_k) > \hat{\tau} \Leftrightarrow \zeta'(d_k) < 0$, the relationship between d_k and c_k (in Figure 7) is inverted with respect to the preceding case under regime $\mathcal{R}_2$ (see Figure 4).

Second, the locus $\dot{d}_k / d_k = 0$ now describes an increasing curve, namely

$$\gamma = (\zeta(d_k) + \alpha) y_k(c_k) - c_k = c_k[(\zeta(d_k) + \alpha) c_k^{\psi-1} - 1] = 0.$$

As $\psi > 1$, an increase in c_k rises the economic growth rate, hence a monotonic increasing relationship between c_k and d_k, because $\zeta'(d_k) < 0$.

Therefore, both loci are reversed relative to section 4. Effectively, if $\tau' > \hat{\tau}$, any increase of the debt ratio (d_k) rises public spending (g_k) and thus reduces the economic growth rate (γ), whereas in section 4 with a constant tax rate (or more largely when $\tau' < \hat{\tau}$), public spending and deficit were negatively related (see Eq. (13)). Consequently, by implementing highly debt-sensitive tax rules, the relationship between public debt and economic growth reverses.

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[18]We build the phase portrait for a linear fiscal rule (see Appendix E for technical details).
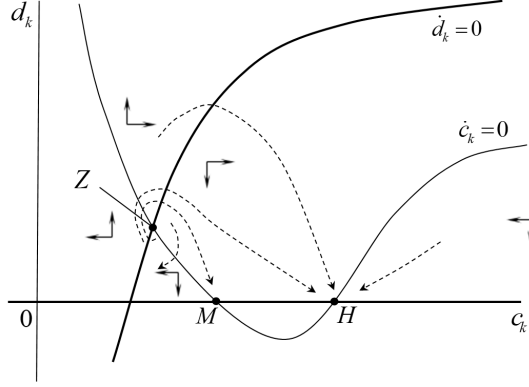
Figure 7: The phase portrait in regime $\mathcal{R}_2$ with $\tau' > \hat{\tau}$

Local dynamics. Let us now study the local stability when $\tau' > \hat{\tau}$. By differentiating system (27), the components of the jacobian matrix $\mathbf{J}^i$ for steady-state $i \in \{H, M, Z\}$, are

$$CC^i = c_k^i - \zeta(d_k^i)\psi y_k^i, \tag{29}$$
$$CD^i = -\zeta'(d_k^i)c_k^i y_k^i, \tag{30}$$
$$DD^i = -\gamma^i - \zeta'(d_k^i)d_k^i y_k^i, \tag{31}$$
$$DC^i = [1 - A\psi(c_k^i)^{\psi-1}(\zeta(d_k^i) + \alpha)]d_k^i. \tag{32}$$

Thus, the trace and the determinant of the Jacobian matrix are, respectively

$$\mathrm{Tr}(\mathbf{J}^i) = c_k^i - \zeta(d_k^i)\psi y_k^i - \gamma^i - \zeta'(d_k^i)d_k^i y_k^i, \tag{33}$$
$$\det(\mathbf{J}^i) = -[c_k^i - \zeta(d_k^i)\psi y_k^i]\gamma^i - \alpha\psi\zeta'(d_k^i)(y_k^i)^2 d_k^i, \tag{34}$$

and the following theorem characterizes the topological behaviour of each BGP.

Theorem 4. *The zero-growth solution (Z) is unstable, the medium-growth solution (M) is saddle-path stable, and the high-growth solution (H) is stable.*

Proof. See Appendix F.

The intuition driving the local behaviour of the positive growth solutions is the same as previously in regime $\mathcal{R}_2$. At points M and H, the two eigenvalues are: $\lambda_1^i = DD^i = -\gamma^i$, and $\lambda_2^i = CC^i = c_k^i - \zeta(0)\psi y_k$. For $\psi > 0$, the medium-growth equilibrium M is such that $CC^M > 0$, hence the law of motion of consumption is unstable. Consequently, there is one negative and one positive eigenvalue, and M is saddle-path stable. In contrast, the point H is such that $CC'^H < 0$. The law of motion of consumption is then stable, the two eigenvalues are negative, and H is locally undetermined (stable).

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The heart of the mechanism that makes the zero-growth solution unstable is related to the relationship between public debt and public spending $(\partial g_k / \partial d_k)$. As discussed above, we can distinguish two cases.

(i) If this relation is negatively-sloped $(\tau' < \hat{\tau})$, any increase in the public debt ratio produces a higher net-of-taxes debt burden that requires a cut in public spending. This cut, in turn, generates a higher rate of capital accumulation in the goods market equilibrium, such that the debt-to-output ratio will decrease. Hence the stability of the debt path, which leads to the saddle-path property of the no-growth solution Z, as in the previous section.

(ii) In the case of a positively-sloped relation $(\tau' > \hat{\tau})$, however, the increase of the debt ratio generates enough fiscal resources too increase public spending despite the BBR. Consequently, the rate of capital accumulation falls in goods market equilibrium, and the debt-to-output ratio undertakes an unstable trajectory. As a result, the no-growth solution becomes unstable.

Global dynamics. If $\tau' > \hat{\tau}$, the economy is characterized by both local indeterminacy (in the vicinity of the high growth solution H) and global indeterminacy. For a given (predetermined) public debt ratio $d_{k0} > 0$, the initial consumption ratio can jump on the unique transition path that converges towards the medium-growth equilibrium (M), or on one of the multiple paths that converge towards H.

The main interest of a varying tax rate is that the economy cannot experiment the no-growth trap in the long run, provided that $\tau' > \hat{\tau}$. This analysis points out the need for indebted countries to implement stringent fiscal rules in order to make tolerable their public finance stance. This feature is consistent with the concept of the "fiscal space", whereby public debt sustainability is ensured only if the primary budget surplus positively reacts to the debt burden (Ostry et al., 2015). As outlined by the proponents of fiscal space, the adjustment of the primary surplus can be made difficult given the presence of some social resistance to tax-increases, notably in a context of high debt. Therefore, the case with $\tau' < \hat{\tau}$ reflects a *narrow fiscal space*, while the case $\tau' > \hat{\tau}$ can be associated to a *large fiscal space*. In these lines, our model reveals that the no-growth trap can be removed only if the fiscal space is large enough.[19] However, as in the previous section, global indeterminacy remains and raises a similar selection problem, since the economy can converge towards different positive growth solutions.

7. Concluding remarks

This paper shows that local and global indeterminacy can appear in a one-sector endogenous growth model, when wasteful public expenditures adjust to the government's

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[19]A related analysis in a two-period game theoretic approach with tax policy becoming procyclical at high debt ratios is conducted by Camous and Gimber (2018).

budget constraint. This result contrasts with usual findings in neoclassical (exogenous) growth models, showing that indeterminacy can emerge only with a tax-based adjustement (Schmitt-Grohé and Uribe, 1997), or when public expenditures are useful, i.e. produce utility services and/or productive externalities (see, e.g. Guo and Harrison, 2008). Therefore, endogenous growth models can reverse results obtained in exogenous growth setups, as in Guo and Harrison (2004).

Fundamentally, this result comes from the nature of the BBR, which is consistent with non trivial dynamics of the debt ratio when considering endogenous growth. In our framework, ignoring negative growth solutions in steady-state, two admissible long-run equilibria emerges: a zero-growth BGP with positive debt, and a positive-growth BGP with zero debt. Both BGPs can be reached, as the stability analysis does not allow selecting a unique equilibrium.

However, a fiscal rule such that the tax rate strongly responds to public-debt increases can remove the no-growth trap, and secure positive long-run growth (even if indeterminacy remains, because two positive BGPs are feasible). This analysis calls for implementing tight tax policies in indebted countries,

This paper can be extended in several directions. For example, it would be worthwhile to examine alternative fiscal rules (see, e.g. Combes et al., 2017), or to introduce the case for money-financing (Menuet et al., 2018). In addition, possible extensions would study the robustness of our theoretical results, as the specification of non-separable preferences, progressive endogenous taxes, or useful public spending. We plan to pursue these research projects in the future.

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Appendix A. Exogenous growth with public debt

In the context of exogenous growth, the production function (8) is rewritten as

$$Y_t = AK_t^\alpha L_t^{1-\alpha},$$

with $\alpha \in (0,1)$, and the model is otherwise unchanged. From Eqs. (6)-(7)-(10)–(11)-(12)-(13), we can write the reduced form of the model. To this end, we use the following logarithmic transformation of variables: $\tilde{k}_t := \log(K_t)$ and $\tilde{c}_t := \log(C_t)$

$$\dot{\tilde{k}}_t = [1 - \tau(1-\alpha) + \alpha D_0 e^{-\tilde{k}_t}]e^{\lambda_0 + \lambda_1 \tilde{k}_t + \lambda_2 \tilde{c}_t} - e^{\tilde{c}_t - \tilde{k}_t}, \tag{A.1}$$

$$\dot{\tilde{c}}_t = \alpha e^{\lambda_0 + \lambda_1 \tilde{k}_t + \lambda_2 \tilde{c}_t} - \rho, \tag{A.2}$$

where $\lambda_0 = (1-\alpha)\log\{(1-\alpha)(1-\tau)/A\}/(\alpha+\varepsilon)$, $\lambda_1 := -\varepsilon(1-\alpha)/(\alpha+\varepsilon)$, and $\lambda_2 := -(1-\alpha)/(\alpha+\varepsilon)$. We thus obtain a dynamic system identical to GH (Eqs 11-12 p. 360, with $\delta = \tau_k = 0$), except that a new term related to public debt (D_0) appears in the dynamics of capital accumulation (A.1).

Lemma 1. *There is a non-empty set of parameters that ensures the existence and uniqueness of the long-run steady-state.*

Proof. In the long-run, from the Keynes-Ramsey relationship ($\dot{C}_t/C_t = \alpha(Y_t/K_t) - \rho = 0$), we have $L/K = (\rho/\alpha A)^{1/(1-\alpha)}$. From the goods market equilibrium (with $\dot{K}_t = 0$), it follows that

$$\frac{C}{K} + \frac{G}{K} = \frac{Y}{K} = \frac{\rho}{\alpha}. \tag{A.3}$$

By the government's budget constraint (10), the public spending ratio is, in steady state (with the real interest rate being $r = \alpha Y/K = \rho$)

$$\frac{G}{K} = \tau(1-\alpha)\frac{\rho}{\alpha} - \rho\frac{D_0}{K}. \tag{A.4}$$

The first-order condition on labor supply (5) now leads to

$$\frac{C}{K} = (1-\tau)(1-\alpha)AK^{-(1+\varepsilon)}\left(\frac{\rho}{\alpha A}\right)^{-(\alpha+\varepsilon)/(1-\alpha)}. \tag{A.5}$$

Introducing (A.5) and (A.4) in (A.3), we obtain an implicit relation in K

$$\mathcal{F}(K) = \mathcal{G}(K),$$

where $\mathcal{F}(K) := A[1 - \tau(1-\alpha)]K$, and $\mathcal{G}(K) := A(1-\tau)(1-\alpha)\left(\frac{\rho}{\alpha A}\right)^{\frac{1-\varepsilon}{1-\alpha}}K^{-\varepsilon} - \alpha AD_0$. Clearly, as $\mathcal{G}$ is a monotonic decreasing continuous function, according to the intermediate value theorem, there is a unique critical value $K^* > 0$ such that $\mathcal{F}(K^*) = \mathcal{G}(K^*)$ for an

usual constellation of parameters. From Eqs. (A.4)-(A.5), we find the associated long-run values of C^* and G^*.

In addition, as $\mathcal{G}(\cdot)$ negatively depends on D_0, any increase in public debt reduces the long-run capital stock (see Figure A.8).
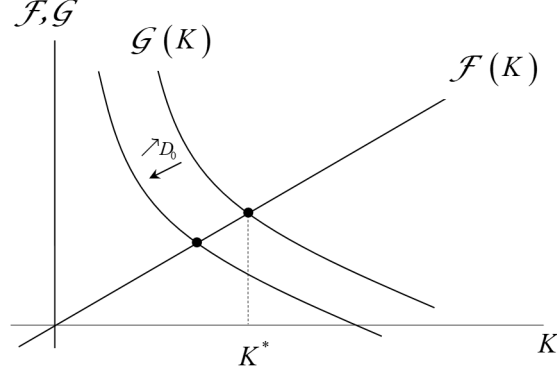

Figure A.8: The steady state

It is straightforward to show that the steady-state is well determined, independently of the public debt level. Effectively, denoting $X := e^{\lambda_0 + \lambda_1 \tilde{k}^* + \lambda_2 \tilde{c}^*}$, the determinant of system (A.1)-(A.2) is, in the neighborhood of the steady state $(\tilde{c}^*, \tilde{k}^*)$,

$$\det = \lambda_1 \alpha X e^{\tilde{c}^* - \tilde{k}^*} + \lambda_2 \alpha X [e^{\tilde{c}^* - \tilde{k}^*} - D_0 X e^{-\tilde{k}^*}] < 0,$$

since, by (A.1), $e^{\tilde{c}^* - \tilde{k}^*} - D_0 X e^{-\tilde{k}^*} = [1 - \tau(1 - \alpha)]X > 0$.

Therefore, in a neoclassical (exogenous) growth model with public spending being the adjustment variable in the government's budget constraint, a constant public debt D_0 reduces the long-run capital stock K^*, without generating multiplicity of long-run solutions or indeterminacy of the transition path. This result generalizes GH's findings to balanced-budget rules associated with a positive debt level.

Appendix B. Proof of Theorem 1

- *Zero-growth solution.* From the Keynes-Ramsey relation (11), we have: $\gamma = \dot{C}/C = 0 \Leftrightarrow \alpha y_k = \rho$. Consequently, the zero-growth solution (denoted by point Z in Figures 3-6) is characterized by $c_k^Z = \left(\frac{\rho}{\alpha A}\right)^{1/\psi} > 0$.

  From (18), $\gamma^Z = 0 \Leftrightarrow (\bar{d} + \alpha + \alpha d_k^Z) y_k^Z - c_k^Z = 0$. Thus, as $c_k^Z > 0$, we have

$$d_k^Z = \frac{1}{\alpha}\left\{\frac{1}{A(c_k^Z)^{\psi-1}} - \bar{d} - \alpha\right\}.$$

The existence of the zero-growth solution (namely $d_k^Z > 0$) is ensured if $\rho < \bar{\rho} := \alpha A[A(\bar{d} + \alpha)]^{\psi/(1-\psi)}$.

- *Non-zero growth solutions.* As $d_k = 0$, we have, using Eq. (17),

$$\Psi(c_k) := c_k - \rho - \bar{d}Ac_k^{\psi} = 0. \tag{B.1}$$

The behaviour of function Ψ depends on the sign of ψ.

(i) $\psi < 0 \Leftrightarrow \phi < 1 + \varepsilon$. In this case, Ψ describes an increasing convex curve for $c_k \in (0, +\infty)$, where $\Psi(0) = -\infty$ and $\Psi(+\infty) = +\infty$. According to the Intermediate Value Theorem, there is a unique critical value $c_k^Q > 0$ such that $\Psi(c_k^Q) = 0$.

Besides, in the long run, the growth rate is $\gamma = \alpha Ac_k^{\psi} - \rho$; hence

$$\gamma \geq 0 \Leftrightarrow \begin{cases} c_k \geq c_k^Z & \text{if } \psi > 1, \\ c_k \leq c_k^Z & \text{if } \psi < 0. \end{cases}$$

Consequently, if $\Psi(c_k^Z) > 0$, i.e. $\rho < \bar{\rho}$, we have $c_k^Q < c_k^Z$, and the growth rate is positive at point $Q = (c_k^Q, 0)$.

(ii) $\psi > 0 \Leftrightarrow \phi > 1 + \varepsilon$. In this case, $\Psi'(c_k) \geq 0 \Leftrightarrow c_k \leq \hat{c}_k := (\psi \bar{d} A)^{1/(1-\psi)}$, with $\Psi(0) = -\rho < 0$, and $\Psi(+\infty) = -\infty$, since $\psi > 0 \Rightarrow \psi > 1$.

Thus, if $\Psi(c_k^Z) > 0$, namely $\rho < \bar{\rho}$, according to the Intermediate Value Theorem, there are two positive solutions: $c_k^N < c_k^Z$, and $c_k^P > c_k^Z$; hence, the economic growth rate is negative at point $N = (c_k^N, 0)$ and positive at point $P = (c_k^P, 0)$.

Appendix C. Proof of Theorem 2

We study the local stability of the steady states in the two regimes.

(i) Zero-growth solution. At point Z, as $\gamma^Z = 0$, we have, from Eq. (26), $\det(\mathbf{J}^Z) = -\psi \alpha^2 (y_k^Z)^2 d_k^Z$.

- If $\psi > 0$ (regime $\mathcal{R}_2$), $\det(\mathbf{J}^Z) < 0$, which means that there are two opposite-sign eigenvalues, and Z is a saddle-point.

- If $\psi < 0$ (regime $\mathcal{R}_1$), $\det(\mathbf{J}^Z) > 0$, and, from Eq. (25), $\text{Tr}(\mathbf{J}^Z) = c_k^Z - \psi \bar{d} y_k^Z - \alpha d_k^Z y_k^Z [\psi + 1] = y_k[(\bar{d} + \alpha) - (\bar{d} + \alpha d_k^Z)\psi] > 0$. Consequently, there are two eigenvalues with positive real part, and Z is unstable (over-determined).

(ii) Non-zero growth solution. As $d_k^i = 0$, from Eq. (26), $\det(\mathbf{J}^i) = -[c_k^i - \bar{d}\psi y_k^i]\gamma^i$, and $\text{Tr}(\mathbf{J}^i) = c_k^i - \bar{d}\psi y_k^i - \gamma^i = -\gamma^i[\det(\mathbf{J}^i) + 1]$; $\forall i \in \{Q, N, P\}$.

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- If $\psi < 0$ (regime $\mathcal{R}_1$), there is a unique non-zero growth solution (point Q). As $\gamma^Q > 0$, we have $\det(\mathbf{J}^Q) < 0$, namely, Q is a saddle-point.

- If $\psi > 0$ (regime $\mathcal{R}_2$), $c_k^i - \bar{d}\psi y_k^i < 0 \Leftrightarrow c_k^i > \hat{c}_k := [1/\bar{d}A\psi]^{1/(\psi-1)}$. As $c_k^N < \hat{c}_k$, and $\gamma^N < 0$ (see Appendix B), we have $\det(\mathbf{J}^N) > 0$, and $\mathrm{Tr}(\mathbf{J}^N) > 0$, namely, N is unstable (over-determined). In contrast, as $c_k^P > \hat{c}_k$ and $\gamma^P > 0$, we have $\det(\mathbf{J}^P) > 0$, and $\mathrm{Tr}(\mathbf{J}^P) < 0$, namely, P is stable (under-determined).

Appendix D. Proof of Theorem 3

Let us define $\zeta(d_k) := \bar{d}(d_k) + \alpha d_k$, with $\bar{d}(d_k) = (1 - \tau(d_k))(1 - \alpha)$. As $\tau'(d_k) \geq \hat{\tau}$, we have $\zeta'(d_k) \leq 0$, $\forall d_k \geq 0$. From system (27), as in section 3, there are two types of solutions: a zero-growth solution ($d_k > 0 \Rightarrow \gamma = 0$), and non-zero growth solutions ($\gamma > (<)0 \Rightarrow d_k = 0$).

(i) Zero growth solution. Following the proof of Theorem 1, the zero-growth BGP (denoted by point Z in Figure 7) is still characterized by $c_k^Z = \left(\frac{\rho}{\alpha A}\right)^{1/\psi} > 0$. Besides, we have

$$\gamma^Z = 0 \Leftrightarrow c_k^Z - \rho - \zeta(d_k^Z)A(c_k^Z)^\psi = 0 \Leftrightarrow \zeta(d_k^Z) = \frac{c_k^Z - \rho}{A(c_k^Z)^\psi}.$$

As ζ is a continuous and monotonic function, there is an inverse function ζ^{-1}, and

$$d_k^Z = \zeta^{-1}\left(\frac{c_k^Z - \rho}{A(c_k^Z)^\psi}\right).$$

As $\zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+$, this solution exists if:
(a) $c_k^Z > \rho$, i.e. $\rho < \hat{\rho}_1 := (\alpha A)^{1/(1-\psi)}$,
(b) $(c_k^Z - \rho)/A(c_k^Z)^\psi < \zeta(0)$, i.e. $\tau(0) < \bar{\tau}_0 := 1 - \frac{1}{1-\alpha}\left(\frac{c_k^Z - \rho}{A(c_k^Z)^\psi}\right)$.

(ii) Non-zero growth solutions. As $d_k = 0$, for $c_k > 0$ we have, using Eq. (27)

$$\Phi(c_k) := c_k - \rho - \zeta(0)Ac_k^\psi = 0,$$

hence, $\Phi'(c_k) > 0 \Leftrightarrow c_k < \hat{c}_k := [A\psi\zeta(0)]^{1/(1-\psi)}$, with $\Phi(0) = -\rho$ and $\Phi(+\infty) = -\infty < 0$. Consequently, if $\Phi(\hat{c}_k) > 0$, i.e. $\rho < \hat{\rho}_2 := \hat{c}_k(\psi - 1)/\psi$, according to the Intermediate Value Theorem, there are two positive solutions: $\underline{c}_k < \hat{c}_k$ and $\bar{c}_k > \hat{c}_k$.

In addition, the condition (b) $\tau(0) < \bar{\tau}_0$ states that $\Phi(c_k^Z) < 0$, namely either $c_k^Z < \underline{c}_k < \hat{c}_k < \bar{c}_k$ or $\underline{c}_k < \hat{c}_k < \bar{c}_k < c_k^Z$. Under the sufficient condition $c_k^Z < \hat{c}_k$, i.e. $\rho < \hat{\rho}_3 := \alpha A[A\psi\zeta(0)]^{\psi/(1-\psi)}$, it is clear that $c_k^Z < \underline{c}_k < \bar{c}_k$.

Therefore, there are three solutions: a zero growth solution (Z), and two positive growth solutions that we denote, respectively, the medium-growth solution (point M) corresponding to $\underline{c}_k =: c_k^M$, and the high-growth solution (point H) corresponding to $\bar{c}_k =: c_k^H$.

Finally, we denote $\hat{\rho} := \min\{\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\}$.

Appendix E. Phase portrait for a linear tax rule (with $\psi > 1$)

Let us consider the linear fiscal rule $\tau(d_k) = \eta d_k + \eta_0$, where η_0 is a constant ensuring $\tau(\cdot) \in (0, 1)$, and $\eta > 0$. From system (27), the steady-state is obtained at the crossing-points of two stationary loci. By denoting $\zeta(d_k) := \bar{d}(d_k) + \alpha d_k$, we have $\zeta(d_k) = d_k[\alpha - \eta(1-\alpha)] + (1 - \eta_0)(1 - \alpha)$, hence $\zeta'(\cdot) > 0 \Leftrightarrow \eta < \hat{\eta} := \alpha/(1-\alpha)$.

The locus $\dot{c}_k = 0$ is described by the following equation

$$d_k = \frac{1}{\alpha - \eta(1-\alpha)} \left\{ \frac{c_k - \rho}{A c_k^\psi} - (1 - \eta_0)(1 - \alpha) \right\} =: f_c(c_k).$$

If $\eta > \hat{\eta}$, we have $f_c \in C^2(\mathbb{R}_+^*)$, with $f_c'(c_k) \geq 0 \Leftrightarrow c_k \geq \tilde{c}_k =: \psi\rho/(\psi - 1) > 0$, $f_c(0) = +\infty$ and $f_c(+\infty) = -(1 - \eta_0)(1 - \alpha)/[\alpha - \eta(1 - \alpha)] > 0$. Thus, f_c describes a U-shaped curve in the (c_k, d_k)-plane (as in Figure 7), and there are two positive growth solutions if and only if $f_c(\tilde{c}_k) < 0$.

The locus $\dot{d}_k = 0$ is such that (considering $c_k > 0$)

$$d_k = 0 \text{ or } d_k = \frac{1}{\alpha - \eta(1-\alpha)} \left\{ \frac{1}{A c_k^{\psi-1}} - (1 - \eta_0)(1 - \alpha) - \alpha \right\} =: f_d(c_k)$$

If $\eta > \hat{\eta}$, we have $f_d \in C^2(\mathbb{R}_+^*)$, with $f_d' > 0$ (as $\psi > 1$), $f_d(0) = -\infty$ and $f_d(+\infty) = -[(1 - \eta_0)(1 - \alpha) + \alpha]/[\alpha - \eta(1 - \alpha)] > f_c(+\infty) > 0$. Thus, f_d describes an increasing curve in the (c_k, d_k)-plane (as in Figure 7). Finally, since $d_k^Z > 0$, the two stationary loci cross each other for positive values of d_k.

Appendix F. Proof of Theorem 4

Let us suppose that $\psi > 0$, and $\tau' > \hat{\tau}$, namely $\zeta'(\cdot) < 0$.

- Zero-growth solution ($\gamma^Z = 0$). In this case, we have: $\det(\mathbf{J}^Z) = -\psi\alpha\zeta'(d_k^Z)(y_k^Z)^2 d_k^Z > 0$, and $\text{Tr}(\mathbf{J}^Z) = c_k^Z - \zeta(d_k^Z)\psi y_k^Z - \zeta'(d_k^Z)d_k^Z y_k^Z > c_k^Z - \zeta(0)\psi y_k^Z$, as ζ is a decreasing function. As $c_k^Z < \tilde{c}_k := [A\psi\zeta(0)]^{1/(1-\psi)}$ (see Appendix D), it follows that $\text{Tr}(\mathbf{J}^Z) > 0$. Consequently, there are two eigenvalues with positive real part, and Z is unstable.

- Positive-growth solutions. In this case, as $d_k^i = 0$, $i \in \{M, H\}$, we have: $\det(\mathbf{J}^i) = -[c_k^i - \zeta(0)\psi y_k^i]\gamma^i$. As $\gamma^M, \gamma^H > 0$, $c_k^M < \tilde{c}_k$ and $c_k^H > \tilde{c}_k$, it is clear that $\det(\mathbf{J}^M) < 0$ and $\det(\mathbf{J}^H) > 0$. Therefore, around steady-state M, there is one negative and one positive eigenvalue, so that M is saddle-point stable. Besides, we have: $\text{Tr}(\mathbf{J}^H) = c_k^H - \zeta(0)\psi y_k^H - \gamma^H = -\gamma^H[\det(\mathbf{J}^H) + 1] < 0$, namely there are two eigenvalues with negative real part, and H is stable.